Homework 8

1. (a) If $\psi_n(x)$ is the solution to the one dimensional harmonic oscillator with energy $\hbar\omega(n+\frac{1}{2})$, then the solutions to this three dimensional harmonic oscillator are $\psi_{n_1}(x)\,\psi_{n_2}(y)\,\psi_{n_3}(z)$ with energy

$$E = \hbar\omega\left(n_1 + n_2 + n_3 + \tfrac{3}{2}\right).$$

(b) The energy of the ground state is

$$E = \hbar\omega\left(\tfrac{3}{2}\right).$$

There is only one state with this energy $(n_1 = n_2 = n_3 = 0)$.

(c) The energy of the first excited state is

$$E = \hbar\omega\left(\tfrac{3}{2} + 1\right) = \hbar\omega\left(\tfrac{5}{2}\right).$$

It has degeneracy 3: $(n_1, n_2, n_3) = (1,0,0)$, $(0,1,0)$, or $(0,0,1)$.

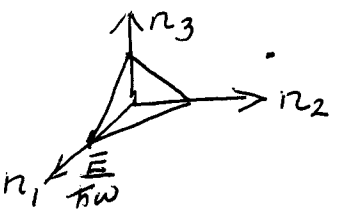
(d) The total number of states with energy less than E is

$$N = \sum_{n_1}\sum_{n_2}\sum_{n_3} 1 \Big|_{E > \hbar\omega(n_1+n_2+n_3+\frac{3}{2})}$$

$$\approx \int_0^\infty dn_1 \int_0^\infty dn_2 \int_0^\infty dn_3\, 1 \Big|_{E > \hbar\omega(n_1+n_2+n_3+\frac{3}{2})}$$

$$\approx \int_0^\infty dn_1 \int_0^\infty dn_2 \int_0^\infty dn_3\, 1 \Big|_{\frac{E}{\hbar\omega} > n_1+n_2+n_3}$$

This is the volume of



I will evaluate this volume in two different ways. First, we do an integral as in Calc. 3:

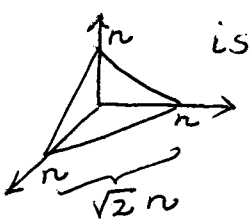
$$N = \int_0^{E/\hbar\omega} dn_1 \int_0^{E/\hbar\omega - n_1} dn_2 \int_0^{E/\hbar\omega - n_1 - n_2} dn_3 \; 1$$

$$= \int_0^{E/\hbar\omega} dn_1 \int_0^{E/\hbar\omega - n_1} dn_2 \left(\frac{E}{\hbar\omega} - n_1 - n_2\right)$$

$$= \int_0^{E/\hbar\omega} dn_1 \left\{\left(\frac{E}{\hbar\omega} - n_1\right)^2 - \frac{1}{2}\left(\frac{E}{\hbar\omega} - n_1\right)^2\right\} = \int_0^{E/\hbar\omega} dn_1 \, \frac{1}{2}\left(n_1 - \frac{E}{\hbar\omega}\right)^2$$

$$= \frac{1}{6}\left(n_1 - \frac{E}{\hbar\omega}\right)^3 \Big|_0^{E/\hbar\omega} = \frac{1}{6}\left(\frac{E}{\hbar\omega}\right)^3 .$$

Second, the area of the triangle is



$$\frac{1}{2}(\sqrt{2}\,n)^2 \sin 60^\circ = n^2 \frac{\sqrt{3}}{2} .$$

This triangle lies in the plane:

$$n_1 + n_2 + n_3 = n \quad \text{or} \quad \left(\frac{\hat{x}}{\sqrt{3}} + \frac{\hat{y}}{\sqrt{3}} + \frac{\hat{z}}{\sqrt{3}}\right)\cdot\left(n_1\hat{x} + n_2\hat{y} + n_3\hat{z}\right) = \frac{n}{\sqrt{3}} .$$

The separation of the planes for n & $n + dn$ is thus $\frac{dn}{\sqrt{3}}$. The volume is

$$N = \int_0^{E/\hbar\omega} n^2 \frac{\sqrt{3}}{2} \frac{dn}{\sqrt{3}} = \frac{1}{6}\left(\frac{E}{\hbar\omega}\right)^3 .$$

In both cases: $N = \frac{1}{6}\left(\frac{E}{\hbar\omega}\right)^3$

& with spin: $\boxed{2N = \frac{1}{3}\left(\frac{E}{\hbar\omega}\right)^3}$

(e) This is similar to (d) except one is summing energy instead of counting states. Using the second method from part (d),

$$E = \hbar\omega\left(n_1 + n_2 + n_3 + \frac{3}{2}\right) = \hbar\omega\left(n + \frac{3}{2}\right),$$

$$E_{tot} = 2\int_0^{E_{max}/\hbar\omega} \hbar\omega\left(n + \frac{3}{2}\right) n^2 \frac{\sqrt{3}}{2}\frac{dn}{\sqrt{3}}$$

for spin

$$= \frac{\hbar\omega}{2} 2\int_0^{E_{max}/\hbar\omega} \left(n^3 + \frac{3}{2}n^2\right) dn$$

$$= \frac{\hbar\omega}{2} 2\left(\frac{1}{4}\left(\frac{E_{max}}{\hbar\omega}\right)^4 + \frac{3}{2}\frac{1}{3}\left(\frac{E_{max}}{\hbar\omega}\right)^3\right)$$

Since $2N = \frac{1}{3}\left(\frac{E_{max}}{\hbar\omega}\right)^3$, $\frac{E_{max}}{\hbar\omega} = (3 \times 2N)^{1/3}$,

$$\boxed{E_{tot} = \hbar\omega\left(\frac{3^{4/3}}{4}(2N)^{4/3} + \frac{3}{2}(2N)\right)}$$

2. We use the identity

$$[AB, C] = ABC - CAB$$
$$- ACB + ACB$$
$$= A[B, C] + [A, C]B.$$

(a) $[L^2, L_z] = [L_x^2, L_z] + [L_y^2, L_z]$
$$= L_x[L_x, L_z] + [L_x, L_z]L_x$$
$$+ L_y[L_y, L_z] + [L_y, L_z]L_y$$
$$= L_x(-i\hbar L_y) + (-i\hbar L_y)L_x$$
$$+ L_y(i\hbar L_x) + (i\hbar L_x)L_y$$
$$= 0$$

(b) $[L_x^2 L_y, L_x] = L_x^2[L_y, L_x] + [L_x^2, L_x]L_y$
$$= L_x^2(-i\hbar L_z) + 0 = -i\hbar L_x^2 L_z$$

(c) $[L_y^3, L_x] = L_y^2[L_y, L_x] + [L_y^2, L_x]L_y$
$$= L_y^2(-i\hbar L_z) + L_y[L_y, L_x]L_y + [L_y, L_x]L_y^2$$
$$= -i\hbar(L_y^2 L_z + L_y L_z L_y + L_z L_y^2)$$

(d) $[L_z^2 L_y, L_x] = L_z^2[L_y, L_x] + [L_z^2, L_x]L_y$
$$= L_z^2(-i\hbar L_z) + L_z[L_z, L_x]L_y + [L_z, L_x]L_z L_y$$
$$= -i\hbar L_z^3 \quad i\hbar L_z L_y^2 \quad + i\hbar L_y L_z L_y$$

(e) $\rightarrow [L^2 L_y, L_x] = -i\hbar(L_x^2 L_z + L_y^2 L_z + L_z^2 L_z)$
$$= -i\hbar L^2 L_z$$

This makes sense because $[L^2, L_\alpha] = 0 \rightarrow$
$[L^2 L_y, L_x] = L^2[L_y, L_x] = L^2(-i\hbar L_z)$.