Homework 9:

1. (a) For $m = l$, the spherical harmonics are

$$Y_0^0(\theta,\varphi) = \sqrt{\frac{1}{4\pi}}$$

$$Y_1^1(\theta,\varphi) = \frac{-1}{2}\sqrt{\frac{6}{4\pi}}\sin\theta\, e^{i\varphi} = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi}$$

$$Y_2^2(\theta,\varphi) = \frac{1}{4\cdot 2}\sqrt{\frac{5!}{4\pi}}\sin^2\theta\, e^{2i\varphi} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{2i\varphi}$$

$$L_- Y_1^1 = \hbar e^{-i\varphi}\left(\frac{-\sqrt{3}}{\sqrt{8\pi}}\right)\left(-\cos\theta + i\cos\theta\, i\right)e^{i\varphi}$$

$$= \hbar\, 2\sqrt{\frac{3}{8\pi}}\cos\theta = \hbar\sqrt{1(1+1) - 1(1-1)}\, Y_1^0 = \hbar\sqrt{2}\, Y_1^0$$

$$\rightarrow Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$L_- Y_1^0 = \hbar e^{-i\varphi}\sqrt{\frac{3}{4\pi}}\left(\sin\theta\right) = \hbar\sqrt{1(1+1) - 0(0-1)}\, Y_1^{-1} = \hbar\sqrt{2}\, Y_1^{-1}$$

$$\rightarrow Y_1^{-1}(\theta,\varphi) = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\varphi}$$

$$L_- Y_2^2 = \hbar e^{-i\varphi}\sqrt{\frac{15}{32\pi}}\left(-2\sin\theta\cos\theta + i\sin\theta\cos\theta\cdot 2i\right)e^{2i\varphi}$$

$$= -\hbar 4\sqrt{\frac{15}{32\pi}}\sin\theta\cos\theta e^{i\varphi}$$

$$= \hbar\sqrt{2(2+1)-2(2-1)}\,Y_2^1 = \hbar 2 Y_2^1$$

$$\rightarrow Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\varphi}$$

$$L_- Y_2^1 = \hbar e^{-i\varphi}\left(-\sqrt{\frac{15}{8\pi}}\right)\left(-\cos^2\theta + \sin^2\theta + i\cos^2\theta\, i\right)e^{i\varphi}$$

$$= \hbar\left(-\sqrt{\frac{15}{8\pi}}\right)\left(-2\cos^2\theta + 1 - \cos^2\theta\right)$$

$$= \hbar\sqrt{2(2+1)-1(1-1)}\,Y_2^0 = \hbar\sqrt{6}\,Y_2^0$$

$$\rightarrow Y_2^0 = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$$

$$L_- Y_2^0 = \hbar e^{-i\varphi}\sqrt{\frac{5}{16\pi}}\left(6\cos\theta\sin\theta\right)$$

$$= \hbar\sqrt{2(2+1)-0(0-1)}\,Y_2^{-1} = \hbar\sqrt{6}\,Y_2^{-1}$$

$$\rightarrow Y_2^{-1} = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{-i\varphi}$$

$$L_- Y_2^{-1} = \hbar e^{-i\varphi}\sqrt{\frac{15}{8\pi}}\left(-\cos^2\theta + \sin^2\theta + i\cos^2\theta(-i)\right)e^{-i\varphi}$$

$$= \hbar\sqrt{\frac{15}{8\pi}}\sin^2\theta\, e^{-2i\varphi}$$

$$= \hbar\sqrt{2(2+1)+1(-1-1)}\; Y_2^{-2} = \hbar 2 Y_2^{-2}$$

$$\rightarrow Y_2^{-2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{-2i\varphi}$$

(b) $$\langle 2,2|2,2\rangle = \langle 2,-2|2,-2\rangle = \frac{15}{32\pi}\int_0^{\pi}\sin\theta\, d\theta\int_0^{2\pi}d\varphi\, \sin^4\theta$$

$$= \frac{15}{32\pi}\cdot 2\pi\int_{-1}^{1}du\,(1-u^2)^2, \text{ where } u=\cos\theta$$

$$= \frac{15}{16}\int_{-1}^{1}du\,(1-2u^2+u^4)$$

$$= \frac{15}{16}\left(u - \frac{2u^3}{3} + \frac{u^5}{5}\right)\Big|_{-1}^{1}$$

$$= \frac{15}{16}2\left(1-\frac{2}{3}+\frac{1}{5}\right) = \frac{15}{8}\left(\frac{15-10+3}{15}\right) = \frac{15}{8}\,\frac{8}{15} = 1 \checkmark$$

$$\langle 2,1|2,1\rangle = \langle 2,-1|2,-1\rangle = \frac{15}{8\pi}\int_0^{\pi}\sin\theta\, d\theta\int_0^{2\pi}d\varphi\, \sin^2\theta\cos^2\theta$$

$$= \frac{15}{8\pi}2\pi\int_{-1}^{1}du\; u^2(1-u^2) = \frac{15}{4}\left(\frac{u^3}{3}-\frac{u^5}{5}\right)\Big|_{-1}^{1}$$

$$= \frac{15}{4}\,2\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{15}{2}\,\frac{5-3}{15} = 1 \checkmark$$

$$\langle 2,0|2,0\rangle = \frac{5}{16\pi}\int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\varphi\,(3\cos^2\theta - 1)^2$$

$$= \frac{5\cdot 2\pi}{16\pi}\int_{-1}^{1} du\,(3u^2-1)^2$$

$$= \frac{5}{8}\int_{-1}^{1} du\,(9u^4 - 6u^2 + 1)$$

$$= \frac{5}{8}\left(\frac{9u^5}{5} - \frac{6u^3}{3} + u\right)\Big|_{-1}^{1} = \frac{5\cdot 2}{8}\left(\frac{9}{5} - 2 + 1\right) = \frac{5}{4}\,\frac{4}{5} = 1 \checkmark$$

(c) $\langle 1,0|1,1\rangle \propto \int_0^{2\pi} d\varphi\, e^{i\varphi} = 0$

$\langle 1,0|1,-1\rangle \propto \int_0^{2\pi} d\varphi\, e^{-i\varphi} = 0$

$\langle 1,-1|1,1\rangle \propto \int_0^{2\pi} d\varphi\, e^{2i\varphi} = 0$

2. $\frac{x}{r} = \sin\theta\cos\varphi$, $\frac{y}{r} = \sin\theta\sin\varphi$, $\frac{z}{r} = \cos\theta$,

and $r = \sqrt{x^2+y^2+z^2}$.

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,1} = -\sqrt{\frac{3}{8\pi}}\ \frac{x+iy}{r}$$

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\ \frac{z}{r}$$

$$Y_{2,2} = \sqrt{\frac{15}{32\pi}}\left(\frac{x+iy}{r}\right)^2 \qquad *$$

$$Y_{2,1} = -\sqrt{\frac{15}{8\pi}}\left(\frac{x+iy}{r}\right)\left(\frac{z}{r}\right)$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}\left(3\left(\frac{z}{r}\right)^2 - 1\right)$$

3. $$\langle Y_{\ell m_1} | L_+ | Y_{\ell m_2}\rangle = \hbar\sqrt{\ell(\ell+1) - m_2(m_2+1)}\,\langle Y_{\ell m_1} | Y_{\ell m_2+1}\rangle$$
$$= \hbar\sqrt{\ell(\ell+1) - m_2(m_2+1)}\ \delta_{m_1, m_2+1}$$

$$\langle Y_{\ell m_1} | L_- | Y_{\ell m_2}\rangle = \hbar\sqrt{\ell(\ell+1) - m_2(m_2-1)}\,\langle Y_{\ell m_1} | Y_{\ell m_2-1}\rangle$$
$$= \hbar\sqrt{\ell(\ell+1) - m_2(m_2-1)}\ \delta_{m_1, m_2-1}$$

Since $L_x = \frac{1}{2}(L_+ + L_-)$ and $L_y = \frac{1}{2i}(L_+ - L_-)$,

$$\langle Y_{\ell m_1} | L_x | Y_{\ell m_2}\rangle = \frac{\hbar}{2}\sqrt{\ell(\ell+1) - m_2(m_2+1)}\ \delta_{m_1, m_2+1} + \frac{\hbar}{2}\sqrt{\ell(\ell+1) - m_2(m_2-1)}\ \delta_{m_1, m_2-1}$$

$$\langle Y_{\ell m_1} | L_y | Y_{\ell m_2}\rangle = \frac{\hbar}{2i}\sqrt{\ell(\ell+1) - m_2(m_2+1)}\ \delta_{m_1, m_2+1} - \frac{\hbar}{2i}\sqrt{\ell(\ell+1) - m_2(m_2-1)}\ \delta_{m_1, m_2-1}$$

$$* \quad e^{2i\phi} = \cos 2\phi + i\sin 2\phi$$
$$= (\cos^2\phi - \sin^2\phi) + i\, 2\sin\phi\cos\phi$$
$$= (\cos\phi + i\sin\phi)^2$$

4. $$L_x^2 = \left(\frac{L_+ + L_-}{2}\right)^2 = \frac{1}{4}L_+^2 + \frac{1}{4}L_-^2 + \frac{1}{4}(L_+L_- + L_-L_+)$$

$$L_y^2 = \left(\frac{L_+ - L_-}{2i}\right)^2 = \frac{-1}{4}L_+^2 - \frac{1}{4}L_-^2 + \frac{1}{4}(L_+L_- + L_-L_+)$$

$$L_x^2 + L_y^2 = \frac{1}{2}(L_+L_- + L_-L_+) = L^2 - L_z^2$$

$$\rightarrow \langle Y_{\ell m_1} | L_x^2 | Y_{\ell m_2} \rangle = \frac{\hbar^2}{2}\left(\ell(\ell+1) - m_2^2\right)\delta_{m_1,m_2}$$
$$+ \frac{\hbar^2}{4}\sqrt{\ell(\ell+1) - m_2(m_2+1)}\sqrt{\ell(\ell+1) - (m_2+1)(m_2+2)}\,\delta_{m_1,m_2+2}$$
$$+ \frac{\hbar^2}{4}\sqrt{\ell(\ell+1) - m_2(m_2-1)}\sqrt{\ell(\ell+1) - (m_2-1)(m_2-2)}\,\delta_{m_1,m_2-2}$$

$$\langle Y_{\ell m_1} | L_y^2 | Y_{\ell m_2} \rangle = \frac{\hbar^2}{2}\left(\ell(\ell+1) - m_2^2\right)\delta_{m_1,m_2}$$
$$- \frac{\hbar^2}{4}\sqrt{\ell(\ell+1) - m_2(m_2+1)}\sqrt{\ell(\ell+1) - (m_2+1)(m_2+2)}\,\delta_{m_1,m_2+2}$$
$$- \frac{\hbar^2}{4}\sqrt{\ell(\ell+1) - m_2(m_2-1)}\sqrt{\ell(\ell+1) - (m_2-1)(m_2-2)}\,\delta_{m_1,m_2-2}$$