Fourier Analysis:

Define the fourier transform of $\psi(x)$ as

$$\psi(k) = \int dx \, e^{-ikx} \psi(x).$$

This is a different convention than the book, but the most common convention in the physics literature. Also note the sloppy phycisist notation for $\psi(k) \neq \psi(x)$.

Theorem: Then $\psi(x) = \int \frac{dk}{2\pi} e^{ikx} \psi(k)$,

or equivalently

$$\psi(x) = \int \frac{dk}{2\pi} e^{ikx} \int dx' \, e^{-ikx'} \psi(x')$$
$$= \int dx' \int \frac{dk}{2\pi} e^{ik(x-x')} \psi(x'),$$

i.e. $\int \frac{dk}{2\pi} e^{ik(x-x')} = \delta(x-x')$

Proof: $$\int_{-N}^{N} \frac{dk}{2\pi} e^{ik(x-x')} = \frac{1}{2\pi} \frac{e^{iN(x-x')} - e^{-iN(x-x')}}{i(x-x')}$$
$$= \frac{\sin(N(x-x'))}{\pi(x-x')}$$

The function $\frac{\sin(N(x-x'))}{\pi(x-x')}$ satisfies $\int_{-\infty}^{+\infty} \frac{\sin(Nx)}{\pi x} dx = 1$ and it becomes infinitely narrow and peaked as $N \to \infty$.

$$\therefore \lim_{N\to\infty} \int_{-N}^{N} \frac{dk}{2\pi} e^{ikx} = \delta(x) \quad //$$

Sometimes we will also use

$$\psi(p) = \int dx\, e^{-i\frac{p}{\hbar}x}\, \psi(x)$$

$$\psi(x) = \int \frac{dp}{2\pi\hbar}\, e^{i\frac{p}{\hbar}x}\, \psi(p) \quad .$$

Normalization of $\psi(p)$:

$$1 = \int dx\, \psi^*(x)\, \psi(x)$$

$$= \int dx \left(\int \frac{dp_1}{2\pi\hbar}\, e^{-i\frac{p_1 x}{\hbar}}\, \psi(p_1)^* \right) \left(\int \frac{dp_2}{2\pi\hbar}\, e^{i\frac{p_2 x}{\hbar}}\, \psi(p_2) \right)$$

$$= \int \frac{dp_1}{2\pi\hbar} \int \frac{dp_2}{2\pi\hbar}\, \psi(p_1)^* \psi(p_2) \int dx\, e^{i\frac{(p_2 - p_1)}{\hbar}x}$$

$$= \int \frac{dp_1}{2\pi\hbar} \int \frac{dp_2}{2\pi\hbar}\, \psi(p_1)^* \psi(p_2)\, \underbrace{2\pi\delta\left(\frac{p_2 - p_1}{\hbar}\right)}_{2\pi\hbar\, \delta(p_2 - p_1)}$$

$$= \int \frac{dp_1}{2\pi\hbar}\, \psi(p_1)^* \psi(p_1)$$

$$= \int \frac{dp_1}{2\pi\hbar}\, |\psi(p_1)|^2$$

Momentum operator in momentum representation:

$$\langle p \rangle = \int dx \; \psi^*(x) \, \frac{\hbar}{i} \frac{d}{dx} \psi(x)$$

$$= \int dx \left(\int \frac{dp_1}{2\pi\hbar} e^{-\frac{i p_1 x}{\hbar}} \psi(p_1)^* \right) \frac{\hbar}{i} \frac{d}{dx} \left(\int \frac{dp_2}{2\pi\hbar} e^{\frac{i p_2 x}{\hbar}} \psi(p_2) \right)$$

$$= \int dx \left(\int \frac{dp_1}{2\pi\hbar} e^{-\frac{i p_1 x}{\hbar}} \psi(p_1)^* \right) p_2 \left(\int \frac{dp_2}{2\pi\hbar} e^{\frac{i p_2 x}{\hbar}} \psi(p_2) \right)$$

$$= \int \frac{dp_1}{2\pi\hbar} \psi(p_1)^* \, p_1 \, \psi(p_1)$$

Position operator in momentum representation:

$$\langle x \rangle = \int dx \; \psi^*(x) \, x \, \psi(x)$$

$$= \int dx \left(\int \frac{dp_1}{2\pi\hbar} e^{-\frac{i p_1 x}{\hbar}} \psi(p_1)^* \right) x \left(\int \frac{dp_2}{2\pi\hbar} e^{i \frac{p_2 x}{\hbar}} \psi(p_2) \right)$$

$$= \int dx \left(\text{''} \qquad \text{''} \right) \frac{\hbar}{i} \frac{d}{dp_2} \left(\text{''} \qquad \text{''} \right)$$

or $$= \int dx \left(\text{''} \qquad \text{''} \right) -\frac{\hbar}{i} \frac{d}{dp_1} \left(\text{''} \qquad \text{''} \right)$$

$$= \int \frac{dp_1}{2\pi\hbar} \int \frac{dp_2}{2\pi\hbar} \psi(p_1)^* \psi(p_2) \, \frac{\hbar}{i} \frac{d}{dp_2} 2\pi\hbar \, \delta(p_1 - p_2)$$

$$= \int \frac{dp_1}{2\pi\hbar} \int \frac{dp_2}{2\pi\hbar} \psi(p_1)^* \left(i\hbar \frac{d\psi(p_2)}{dp_2} \right) 2\pi\hbar \, \delta(p_1 - p_2)$$

$$= \int \frac{dp_2}{2\pi\hbar} \psi(p_2)^* \, i\hbar \frac{d}{dp_2} \psi(p_2)$$