Uncertainty Principle

Commutators:

$$[x,p]\psi(x) = x\frac{\hbar}{i}\frac{d}{dx}\psi(x) - \frac{\hbar}{i}\frac{d}{dx}\,x\,\psi(x)$$

$$= x\frac{\hbar}{i}\frac{d\psi}{dx} - x\frac{\hbar}{i}\frac{d\psi}{dx} - \frac{\hbar}{i}\psi(x)$$

$$= i\hbar\psi(x)$$

$$\Rightarrow \boxed{[x,p]=i\hbar}$$

Expectation values:

$$\langle x\rangle = \int dx\, \psi^*(x)\, x\, \psi(x)$$

$$\delta x = x - \langle x\rangle$$

$$\langle(\delta x)^2\rangle = \int dx\, \psi^*(x)(x-\langle x\rangle)^2\psi(x)$$

$$= \int dx\, \psi^*(x)(x^2 - 2x\langle x\rangle + \langle x\rangle^2)\psi(x)$$

$$= \langle x^2\rangle - 2\langle x\rangle^2 + \langle x\rangle^2$$

$$\langle(\delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$$

$$\Delta x \equiv \sqrt{\langle(\delta x)^2\rangle} = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

Similarly,

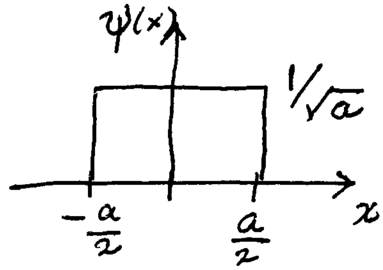
$$\langle p\rangle = \int dx\, \psi^*(x)\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2\psi(x)$$

$$\Delta p \equiv \sqrt{(\delta p)^2} = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$$

Claim: $\Delta x \Delta p \geq \frac{\hbar}{2}$

$\Delta x \Delta k \geq \frac{1}{2}$ $\qquad (p = \hbar k)$

Example:

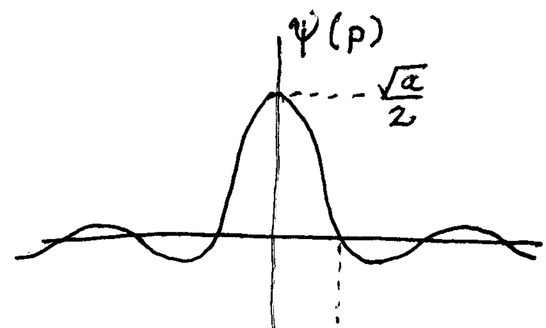


$$\int dx\, |\psi(x)|^2 = 1$$

$$\psi(p) = \int dx\, e^{-i\frac{p}{\hbar}x}\,\psi(x)$$

$$= \frac{e^{-i\frac{p}{\hbar}\frac{a}{2}} - e^{i\frac{p}{\hbar}\frac{a}{2}}}{-ip/\hbar}\,\frac{1}{\sqrt{a}}\quad\frac{a/2}{a/2}$$

$$= \frac{\sqrt{a}}{2}\;\frac{\sin\left(p/\hbar \times \frac{a}{2}\right)}{\frac{pa}{2\hbar}}$$



$\frac{p}{\hbar}\frac{a}{2} = \pi$

$\rightarrow pa = h$

Roughestimate:

$\Delta x \sim a$

$\Delta p \sim 2\frac{h}{a}$

$\Delta x \Delta p \sim 2h.$

<u>Proof</u>: Take an arbitrary $\psi(x)$.

$$\langle x \rangle = \int dx\ \psi^*(x)\, x\, \psi(x)$$

$$\langle p \rangle = \int dx\ \psi^*(x) \frac{\hbar}{i}\frac{d}{dx}\psi(x)$$

$$\delta x = x - \langle x \rangle$$
$$\delta p = p - \langle p \rangle = \frac{\hbar}{i}\frac{d}{dx} - \langle p \rangle$$

Define $\varphi(x) = (\delta x + i\lambda\,\delta p)\,\psi(x)$, where $\lambda \in \mathbb{R}$

$$= \left((x - \langle x \rangle + i\lambda\left(\frac{\hbar}{i}\frac{d}{dx} - \langle p \rangle\right)\right)\psi(x)$$

$$\varphi^*(x) = \left((x - \langle x \rangle - i\lambda\left(-\frac{\hbar}{i}\frac{d}{dx} - \langle p \rangle\right)\right)\psi^*(x)$$

$$0 \leq \int dx\, \varphi^*(x)\varphi(x) = \int dx\ \psi^*(x)\left((x - \langle x \rangle - i\lambda\left(-\frac{\hbar}{i}\overleftarrow{\frac{d}{dx}} - \langle p \rangle\right)\right)$$
$$\left((x - \langle x \rangle + i\lambda\left(\frac{\hbar}{i}\frac{d}{dx} - \langle p \rangle\right)\right)\psi(x)$$

$$= \int dx\ \psi^*(x)\left((x - \langle x \rangle - i\lambda\left(\frac{\hbar}{i}\frac{d}{dx} - \langle p \rangle\right)\right)$$
$$\left((x - \langle x \rangle + i\lambda\left(\frac{\hbar}{i}\frac{d}{dx} - \langle p \rangle\right)\right)\psi(x)$$

$$= \int dx\ \psi^*(x)(\delta x - i\lambda\,\delta p)(\delta x + i\lambda\,\delta p)\psi(x)$$

$$= \langle(\delta x)^2\rangle + \lambda^2\langle(\delta p)^2\rangle - i\lambda\langle[\delta p, \delta x]\rangle$$

$$= (\Delta x)^2 + \lambda^2 \langle (\Delta p)^2 \underbrace{-i\lambda(-i\hbar)}_{-\lambda\hbar}$$

$$\rightarrow (\Delta x)^2 + \lambda^2(\Delta p)^2 - \lambda\hbar \geq 0 \text{ for all } \lambda \in \mathbb{R}$$

Complete the square:

$$\rightarrow (\Delta p)^2\left(\lambda^2 - \frac{\lambda\hbar}{(\Delta p)^2} + \frac{\hbar^2}{4(\Delta p)^4}\right) - \frac{\hbar^2}{4(\Delta p)^2} + (\Delta x)^2 \geqslant 0$$

$$= (\Delta p)^2 \underbrace{\left(\lambda - \frac{\hbar}{2(\Delta p)^2}\right)^2}_{\substack{\text{minimum} = 0 \\ \text{at } \lambda = \hbar/2(\Delta p)^2}} + (\Delta x)^2 - \frac{\hbar^2}{4(\Delta p)^2} \geqslant 0$$

At the minimum:

$$(\Delta x)^2 - \frac{\hbar^2}{4(\Delta p)^2} \geqslant 0 \quad \rightarrow (\Delta x)^2(\Delta p)^2 \geq \frac{\hbar^2}{4}$$

$$\rightarrow \Delta x \, \Delta p \geqslant \frac{\hbar}{2} \quad //$$