Wave Packets:

Consider $\quad i\hbar \dfrac{\partial \psi}{\partial t} = -\dfrac{\hbar^2}{2m}\dfrac{\partial^2 \psi}{\partial x^2}$.

This has solutions

$$\begin{aligned} \psi(x,t) &\propto e^{i(kx-\omega t)} & \text{right moving} \\ &\propto e^{i(kx+\omega t)} & \text{left moving.} \end{aligned}$$

Check: $\quad i\hbar \dfrac{\partial}{\partial t} e^{i(kx-\omega t)} = \hbar\omega\, e^{i(kx-\omega t)}$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} e^{i(kx-\omega t)} = \frac{\hbar^2 k^2}{2m} e^{i(kx-\omega t)}$$

$$\longrightarrow \hbar\omega = \frac{\hbar^2 k^2}{2m} \quad \checkmark$$

$|\psi|^2 \propto |e^{i(kx-\omega t)}|^2 = 1$ has no time dependence.

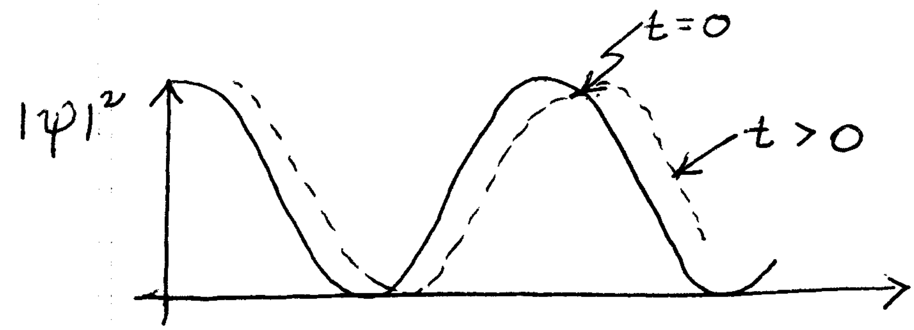
To get time dependence we must add states of different energies.

Let $\quad \psi_1(x,t) = e^{i(k_1 x - \omega_1 t)}$, where $\hbar\omega_1 = \dfrac{\hbar^2 k_1^2}{2m}$

and $\quad \psi_2(x,t) = e^{i(k_2 x - \omega_2 t)}$, where $\hbar\omega_2 = \dfrac{\hbar^2 k_2^2}{2m}$.

$$\underbrace{|\psi_1(x,t) + \psi_2(x,t)|^2}_{\text{also a solution}} = 1 + 1 + e^{i(k_1-k_2)x} e^{-i(\omega_1-\omega_2)t} + e^{i(k_2-k_1)x} e^{-i(\omega_2-\omega_1)t}$$

$$= 2 + 2\cos\big((k_1-k_2)x - (\omega_1-\omega_2)t\big)$$



(group) velocity $= \dfrac{\omega_1 - \omega_2}{k_1 - k_1}$

This moves to the right but is not normalized. Why stop at adding two plane waves:

$$\psi(x,t) = \int \frac{dk}{2\pi} g(k)\, e^{i(kx - \omega_k t)} \quad , \hbar\omega_k = \frac{\hbar^2 k^2}{2m} .$$

This is also a solution. You may wish to write i as:

$$\psi(x,t) = \int \frac{dp}{2\pi\hbar} \psi(p,0)\, e^{i(\frac{p}{\hbar}x - \omega_p t)}, \text{ where}$$

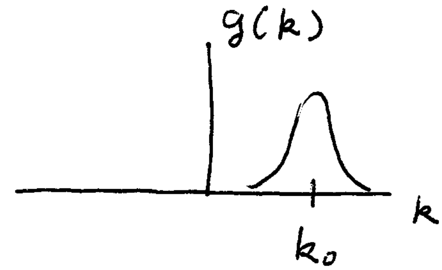
$$\left.\begin{aligned} \psi(x,0) &= \int \frac{dp}{2\pi\hbar} \psi(p,0)\, e^{i\frac{p}{\hbar}x} \\ \psi(p,0) &= \int dx\, \psi(x,0)\, e^{-i\frac{p}{\hbar}x} \end{aligned}\right\} \text{Fourier relations}$$

Normalization:

$$1 = \int dx\, |\psi(x,t)|^2 = \int \frac{dp}{2\pi\hbar} |\psi(p,0) e^{-i\omega_p t}|^2$$
$$= \int \frac{dp}{2\pi\hbar} |\psi(p,0)|^2$$

An example:

$$g(k) = \frac{\sqrt{2}}{(2\pi\alpha)^{1/4}} e^{-\alpha(k-k_0)^2}$$



$$|\psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}} \frac{\alpha^{1/2}}{\sqrt{\alpha^2 + \left(\frac{\hbar t}{2m}\right)^2}} \exp\left(-\frac{\alpha\left(x - \frac{\hbar k_0}{m} t\right)^2}{2\left(\alpha^2 + \left(\frac{\hbar t}{2m}\right)^2\right)}\right).$$

See last year's lecture 3 for details.

This wave packet moves with velocity $\hbar k_0/m$, and the width changes with time.