Eigenvalues & Eigenvectors:

Operator: $f \longrightarrow$ another function

Examples: $\mathcal{O}f(x) = f(x) + x^2$ ... linear op.

$\mathcal{O}f(x) = [f(x)]^2$

$\mathcal{O}f(x) = \frac{df}{dx}$ ... linear op.

Linear operator:

$$L[f_1(x) + f_2(x)] = L[f_1(x)] + L[f_2(x)]$$

$$L[cf(x)] = cL[f(x)]$$

Eigenvalues & eigenvectors:

$$L[f(x)] = \lambda f(x)$$

eigenvector eigenvalue

The operator $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$

is called the hamiltonian. It is linear:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}(\psi_1 + \psi_2) + V(\psi_1 + \psi_2) =$$

$$= \left(-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + V\psi_1\right) + \left(-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} + V\psi_2\right)$$

and $-\frac{\hbar^2}{2m}\frac{d^2 c\psi}{dx^2} + Vc\psi = c\left(-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi\right)$.

The expectation value H for any ψ is real:

$$\langle H \rangle = \int dx\, \psi^*(x)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi(x)$$

$$\langle H \rangle^* = \int dx\, \psi(x)\left(-\frac{\hbar^2}{2m}\frac{d^2\psi^*}{dx} + V\psi^*\right)$$

(integ. by parts twice)

$$= \int dx\, \psi^*(x) -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \psi^* V \psi$$

$$= \langle H \rangle$$

(Such an operator whose expectation value is alway real is called a hermition operator.)

The eigenvalues of a hermition operator are real:

Suppose $H\psi = \lambda\psi$. Then

$$\langle H \rangle = \int dx\, \psi^*(x)\, H\, \psi(x)$$

$$= \lambda \int dx\, \psi^*(x)\psi(x)$$

$$= \text{real}$$

$\Rightarrow \lambda$ is real.

Time independent Schrodinger Eq:

$$E\,\psi(x) = \frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V\psi(x)$$

eigenvector

eigenvalue

Time dependent Schrodinger Eq:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\,\psi(x,t)$$

$\psi(x,t) = \Psi(x)\, e^{-i\frac{E}{\hbar}t}$ is a solution.

Suppose there are many solutions of the time independent Schrodinger Eq.

$$E_n\psi_n = \frac{-\hbar^2}{2m}\frac{d^2\psi_n}{dx^2} + V\psi_n$$

$$\rightarrow \quad i\hbar\frac{\partial}{\partial t}\left(\psi_n e^{-i\frac{E_n}{\hbar}t}\right) = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_n e^{-\frac{iE_n}{\hbar}t} + V\psi_n e^{-\frac{iE_n}{\hbar}t}$$

$$\rightarrow i\hbar\frac{\partial}{\partial t}\sum_n C_n\psi_n e^{-\frac{iE_n}{\hbar}t} = \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right)\sum_n C_n\psi_n e^{-\frac{iE_n}{\hbar}t}$$

$\rightarrow \psi(x,t) = \sum_n C_n\psi_n(x)\, e^{-\frac{iE_n}{\hbar}t}$ is a solution

to the time dependent Schrodinger equation. (C_n are complex constants.)