Hermitian conjugate:

Linear operators:

$$A(\alpha_1 \psi_1(x) + \alpha_2 \psi_2(x)) = \alpha_1 A\psi_1(x) + \alpha_2 A\psi_2(x)$$

Hermitian conjugate:

$$\int dx\, \phi^*(A\psi) = \int dx\, (A^\dagger \phi)^* \psi$$

Suppose $\langle A\rangle$ is real for all ψ. Then $\langle A\rangle = \langle A\rangle^*$

$$\langle A\rangle = \int dx\, \psi^*(A\psi) = \int dx\, (A^\dagger\psi)^* \psi$$

$$\langle A\rangle^* = \int dx\, \psi (A\psi)^* .$$

$\langle A\rangle = \langle A\rangle^*$ if $A = A^\dagger$ (Hermitian operator).

To prove this is a necessary condition consider the wavefunction $\phi + \lambda\psi$.

$$\langle A\rangle = \int dx\, (\phi^* + \lambda^*\psi^*) A(\phi + \lambda\psi)$$

$$= \int dx\, \phi^*(A\phi) + |\lambda|^2 \int dx\, \psi^*(A\psi)$$

$$+ \lambda \int dx\, \phi^*(A\psi) + \lambda^* \int dx\, \psi^*(A\phi)$$

$$\langle A\rangle^* = \int dx\, (\phi + \lambda\psi)((A\phi)^* + \lambda^*(A\psi)^*)$$

$$= \int dx\, \phi\, (A\phi)^* + |\lambda|^2 \int dx\, \psi\, (A\psi)^*$$

$$+ \lambda \int dx\, \psi (A\phi)^* + \lambda^* \int dx\, \phi (A\psi)^*$$

For this to be true for an arbitrary λ, we must have

$$\int dx\, \phi^* (A\psi) = \int dx\, (A\phi)^* \psi \qquad \ldots \lambda \text{ coeff.}$$

$$\int dx\, \psi^* (A\phi) = \int dx\, (A\psi)^* \phi \qquad \ldots \lambda^* \text{ coeff.}$$

These are both the condition that $A = A^\dagger$.

Properties of the Hermitian conjugate:

$$\int dx\, \phi^* (A(B\psi)) = \int dx\, (A^\dagger \phi)^* B\psi = \int dx (B^\dagger A^\dagger \phi)^* \psi$$

$$\Rightarrow (AB)^\dagger = B^\dagger A^\dagger$$

$$\int dx\, \phi^* c A \psi = \int dx\, (c^* A^\dagger \phi)^* \psi$$

$$\Rightarrow (cA)^\dagger = c^* A$$

$\int dx\, \phi^* (A\psi) = \int dx\, (A^\dagger \phi)^* \psi$. Take the complex conjugate:

$$\int dx\, (A\psi)^* \phi = \int dx\, \psi^* (A^\dagger \phi)$$

$$\Rightarrow (A^\dagger)^\dagger = A.$$

More on Dirac Notation:

$$\langle \phi | \psi \rangle = \int dx \, \phi^* \psi$$

$$\langle \phi | \psi \rangle^* = \int dx \, \psi^* \phi = \langle \psi | \phi \rangle$$

$$\langle \phi | A \psi \rangle = \int dx \, \phi^* A \psi \equiv \langle \phi | A | \psi \rangle$$

$$\langle A\phi | \psi \rangle \equiv \int dx \, (A^\dagger \phi)^* \psi \equiv \langle \phi | A^\dagger | \psi \rangle$$

$$\langle \phi | a \psi \rangle = a \langle \phi | \psi \rangle$$

$$\langle a\phi | \psi \rangle = a^* \langle \phi | \psi \rangle$$

Let $\{|a\rangle\}$ be a complete orthonormal basis.

$$|\psi\rangle = \sum_a C_a |a\rangle$$

$$\langle b | \psi \rangle = \sum_a C_a \langle b | a \rangle = \sum_a C_a \delta_{ab} = C_b$$

$$\rightarrow |\psi\rangle = \sum_a \langle a | \psi \rangle \, |a\rangle$$

$$\langle \phi | \psi \rangle = \sum_a \langle a | \psi \rangle \langle \phi | a \rangle$$

$$= \sum_a \langle \phi | a \rangle \langle a | \psi \rangle$$

$$= \langle \phi | \sum_a |a\rangle\langle a| \, |\psi\rangle$$

$$\mathbb{1} = \sum_a |a\rangle\langle a| .$$

Suppose A is Hermitian and

$$A|a\rangle = a|a\rangle$$
$$A|b\rangle = b|b\rangle$$

real eigenvalue

Then $\langle b|A|a\rangle = a\langle b|a\rangle$
$= \langle b|A^{\dagger}|a\rangle = \langle A(b)|a\rangle = b^{*}\langle b|a\rangle = b\langle b|a\rangle$

$\rightarrow (a-b)\langle b|a\rangle = 0.$

If $a \neq b$ (eigenvalue), then $\langle b|a\rangle = 0$, i.e. eigenvectors with different eigenvalues are orthogonal.