Eigenvalues of Angular Momentum

$[J^2, J_\alpha] = 0$ for $\alpha = x, y, z$. $\rightarrow$ We can find simultaneous eigenvectors of J^2 and J_α; however, since $[J_\alpha, J_\beta] \neq 0$ for $\alpha \neq \beta$, we can not find simultaneous eigenvectors of J_x and J_z or J_y and J_z or J_x and J_y. Thus, we choose J^2 and J_z.

Assume: $$J^2 |\psi\rangle = \hbar^2 \lambda |\psi\rangle$$
$$J_z |\psi\rangle = \hbar \lambda_z |\psi\rangle$$

(1) $[J_z, J_+] |\psi\rangle = \hbar J_+ |\psi\rangle$
$= J_z J_+ |\psi\rangle - J_+ J_z |\psi\rangle$
$= J_z (J_+ |\psi\rangle) - \hbar \lambda_z (J_+ |\psi\rangle)$

$\rightarrow J_z (J_+ |\psi\rangle) = \hbar (\lambda_z + 1)(J_+ |\psi\rangle)$

Thus, if $J_z |\psi\rangle = \hbar \lambda_z |\psi\rangle$, then $J_+ |\psi\rangle$ either is zero or has eigenvalue $\hbar(\lambda_z + 1)$.

(2) $[J_z, J_-] |\psi\rangle = -\hbar J_- |\psi\rangle$
$\rightarrow J_z (J_- |\psi\rangle) = \hbar (\lambda_z - 1)(J_- |\psi\rangle)$

Thus, if $J_z |\psi\rangle = \hbar \lambda_z |\psi\rangle$, then $J_- |\psi\rangle$ either is zero or has eigenvalue $\hbar(\lambda_z - 1)$.

(3) Since $[J^2, J_+] = 0$ and $[J^2, J_-] = 0$,

$$J^2(J_+|\psi\rangle) = J_+ J^2 |\psi\rangle = \hbar^2\lambda(J_+|\psi\rangle)$$
$$J^2(J_-|\psi\rangle) = J_- J^2 |\psi\rangle = \hbar^2\lambda(J_-|\psi\rangle).$$

In other words, $J_+|\psi\rangle$ and $J_-|\psi\rangle$ have the same eigenvalue of J^2 as does $|\psi\rangle$. The operators J_+ and J_- do not change the eigenvalue of J^2.

(4) Since $(J_+)^\dagger = J_-$ and $(J_-)^\dagger = J_+$,

$$\begin{aligned} 0 \leq \langle\psi|J_- J_+|\psi\rangle &= \langle\psi|J^2 - J_z^2 - \hbar J_z|\psi\rangle \\ &= \hbar^2\lambda - \hbar^2\lambda_z^2 - \hbar^2\lambda_z \\ &= \hbar^2(\lambda - \lambda_z^2 - \lambda_z) \end{aligned}$$

$$\begin{aligned} 0 \leq \langle\psi|J_+ J_-|\psi\rangle &= \langle\psi|J^2 - J_z^2 + \hbar J_z|\psi\rangle \\ &= \hbar^2\lambda - \hbar^2\lambda_z^2 + \hbar^2\lambda_z \\ &= \hbar^2(\lambda - \lambda_z^2 + \lambda_z) \end{aligned}$$

Since $\lambda - \lambda_z^2 - \lambda_z \geq 0$ and $\lambda - \lambda_z^2 + \lambda_z \geq 0$, $\lambda - \lambda_z^2 \geq 0$. $\rightarrow$ λ_z is bounded from above and below, and $\lambda \geq 0$.

(5) At λ_z^{max} $J_+|\psi\rangle = 0 \rightarrow 0 = \lambda - (\lambda_z^{max})^2 - \lambda_z^{max}$

At λ_z^{min} $J_-|\psi\rangle = 0 \rightarrow 0 = \lambda - (\lambda_z^{min})^2 + \lambda_z^{min}$

These have solutions : $\lambda_z^{max} = \dfrac{-1 \pm \sqrt{1+4\lambda}}{2}$

$$\lambda_z^{min} = \frac{1 \pm \sqrt{1+4\lambda}}{2}$$

Order these from largest to smallest:

largest: $\frac{1}{2} + \frac{\sqrt{1+4\lambda}}{2}$ possible λ_z^{min}

$-\frac{1}{2} + \frac{\sqrt{1+4\lambda}}{2}$ possible λ_z^{max}

$\frac{1}{2} - \frac{\sqrt{1+4\lambda}}{2}$ possible λ_z^{min}

Smallest: $-\frac{1}{2} - \frac{\sqrt{1+4\lambda}}{2}$ possible λ_z^{max}

Since $\lambda_z^{max} \geq \lambda_z^{min}$, $\lambda_z^{max} = -\frac{1}{2} + \frac{\sqrt{1+4\lambda}}{2}$

$\lambda_z^{min} = \frac{1}{2} - \frac{\sqrt{1+4\lambda}}{2}$

This shows that there is only one λ_z for which $J_+|\psi\rangle = 0$, and one λ_z for which $J_-|\psi\rangle = 0$.

(6) If we start at λ_z^{min} and apply J_+ repeatedly, we must eventually end at λ_z^{max}. $\rightarrow$

$$\lambda_z^{max} = \lambda_z^{min} + N \quad \text{, where } N \text{ is an integer } \geq 0.$$

$$0 = \lambda - (\lambda_z^{max})^2 - \lambda_z^{max}$$
$$0 = \lambda - (\lambda_z^{min})^2 + \lambda_z^{min} \rightarrow 0 = \lambda - (\lambda_z^{max} - N)^2 + (\lambda_z^{max} - N)$$

$$\rightarrow \quad 0 = 2N\lambda_z^{max} - N^2 + 2\lambda_z^{max} - N$$
$$= 2(N+1)\lambda_z^{max} - N(N+1)$$

$$\rightarrow \quad \lambda_z^{max} = N/2$$
$$\lambda_z^{min} = \lambda_z^{max} - N = -N/2$$
$$\lambda = \lambda_z^{max}(1 + \lambda_z^{max}) = \frac{N}{2}\left(1 + \frac{N}{2}\right)$$

Check: $\lambda_z = -\frac{1}{2} + \frac{\sqrt{1+4\lambda}}{2} = -\frac{1}{2} + \frac{\sqrt{1+N(2+N)}}{2} = -\frac{1}{2} + \frac{N+1}{2} = \frac{N}{2}$ ✓

Thus, the allowed eigenvalues of J^2 are $\hbar^2 \frac{N}{2}\left(1+\frac{N}{2}\right)$, where $N = 0, 1, 2, \cdots$, and for a given N the eigenvalues of J_z are $-\hbar\frac{N}{2}, -\hbar\frac{N}{2}+\hbar, \ldots, \hbar\frac{N}{2}-\hbar, \hbar\frac{N}{2}$.

This is usually written as

$$J^2 |j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$$
$$J_z |j,m\rangle = \hbar m |j,m\rangle,$$

where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$ and $m = -j, -j+1, \cdots, j-1, j$.

Matrix elements:

$$J^2 |j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$$
$$J_z |j,m\rangle = \hbar m |j,m\rangle$$

$$\rightarrow \langle j,m'| J^2 |j,m\rangle = \hbar^2 j(j+1)\,\delta_{m,m'}$$
$$\langle j,m'| J_z |j,m\rangle = \hbar m\,\delta_{m,m'}$$

$$J^+ |j,m\rangle = C|j,m+1\rangle$$
$$\langle j,m| J^- J^+ |j,m\rangle = |C|^2 \langle j,m+1|j,m+1\rangle = |C|^2$$
$$= \langle j,m| J^2 - J_z^2 - \hbar J_z |j,m\rangle$$
$$= \hbar^2 (j(j+1) - m^2 - m)$$
$$= \hbar^2 (j(j+1) - m(m+1)) \rightarrow C = \hbar\sqrt{j(j+1)-m(m+1)}$$

$$\boxed{J^+ |j,m\rangle = \hbar\sqrt{j(j+1)-m(m+1)}\,|j,m+1\rangle}$$

Check: $J^+|j,m=j\rangle = 0$

$$J^- |j,m\rangle = C|j,m-1\rangle$$
$$\langle j,m| J^+ J^- |j,m\rangle = |C|^2$$
$$= \langle j,m| J^2 - J_z^2 + \hbar J_z |j,m\rangle$$
$$= \hbar^2 (j(j+1) - m^2 + m)$$
$$= \hbar^2 (j(j+1) - m(m-1)) \rightarrow C = \hbar\sqrt{j(j+1)-m(m-1)}$$

$$\boxed{J^- |j,m\rangle = \hbar\sqrt{j(j+1)-m(m-1)}\,|j,m-1\rangle}$$

Check: $J^-|j,m=-j\rangle = 0$