Homework 7 / Practice Exam 2
(due in class on Wed. Oct. 23)

1. **Short answer section**

   (a) Express $\langle\phi|\psi\rangle$ as an integral.

   (b) Express $\langle\phi|p|\psi\rangle$ as an integral.

   (c) For a discrete set of states what is the orthonormality condition in Dirac notation?

   (d) For a discrete set of states what is the completeness condition in Dirac notation?

   (e) What function corresponds to the state $|x_o\rangle$?

   (f) What is the definition of the Hermitian conjugate of an operator A?

   (g) Give two different (but equivalent) definitions of a Hermitian operator.

   (h) What is the Hermitian conjugate of the product of two operators, AB?

   (i) What is the Hermitian conjugate of cA, where c is a complex number?

   (j) If two Hermitian operators commute, $[A, B] = 0$, what can one conclude about the eigenvectors of the operators?

   (k) Assuming that $[A, B] \neq 0$, what is the uncertainty relation for the operators A and B?

   (l) What differential equations do the expectation values of the position and momentum operators satisfy for the Hamiltonian $H = \frac{p^2}{2m} + V(x)$?

   (m) Express the solution of the time-dependent Schrodinger equation at time t in terms of the wave function at $t = 0$ and the eigenvectors of the hamiltonian.

   (n) What is the time evolution operator for a time independent hamiltonian?

   (o) Explain the difference between the Heisenberg and Schrodinger pictures.

   (p) Express the time dependent Heisenberg operator, $A(t)$, in terms of the time independent Schrodinger operator, A.

   (q) What are the eigenvalues of the harmonic oscillator hamiltonian?

   (r) What do the raising and lowering operators do when acting on the n<u>th</u> eigenstate of the harmonic oscillator, $|n\rangle$?

   (s) What is the commutation relation between the the raising and lowering operators?

2. **Operators**

   (a) Prove the identity $[AB, C] = A[B, C] + [A, C]B$ for any three operators A, B, C.

   (b) Using the definition of the Hermitian conjugate, compute the Hermitian conjugate of the operator d/dx.

3. **Principles of quantum mechanics**

The wave functions

$$\phi_n(x) = \frac{e^{ik_n x}}{\sqrt{L}},$$

$k_n = 2\pi n/L$ and $n = 0, \pm 1, \pm 2, \ldots$, form a complete orthonormal basis on the interval $[0, L]$. They are also eigenvectors of the hamiltonian $H = \frac{p^2}{2m}$. Since the wave functions are periodic, $\phi_n(0) = \phi_n(L)$, these wave functions may be thought of as being on a ring with circumference L.

(a) Suppose the wave function at $t = 0$ is

$$\psi(x, 0) = \frac{e^{2\pi i x/L}}{\sqrt{2L}} + i\frac{e^{4\pi i x/L}}{\sqrt{2L}}.$$

Determine the wave function, $\psi(x, t)$, at an arbitrary time.

(b) Compute the probability density at time t.

(c) The probability density is periodic in time. What is the period, τ, of the oscillations?

(d) Sketch the probability density at $\tau = 0$, $\tau/4$, $\tau/2$, and $3\tau/4$.

(e) If the position of the particle is measured at $t = \tau/4$, what is the probability that the particle is between 0 and $L/2$, i.e., $0 \leq x \leq L/2$?

4. **Harmonic oscillator**

(a) At $t = 0$ a harmonic oscillator is in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle.$$

What is the wave function at an arbitrary time t?

(b) For the harmonic oscillator the position and momentum operators may be written as

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a^\dagger + a)$$
$$p = \sqrt{\frac{m\hbar\omega}{2}}\,i\,(a^\dagger - a).$$

Compute the expectation value of the position and momentum as a function of time for the above wave function. Do the results for $\langle x\rangle$ and $\langle p\rangle$ make sense when compared to the classical equations of motion?

(c) Compute the expectation values of x^2 and p^2 as as a function of time. How are these related to the energy?

(d) Finally, using the results from (b) and (c) above, check the uncertainty principle for this wave function.

5. **N-Particle Systems**

Consider two electrons described by the Hamiltonian

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(x_1) + V(x_2),$$

where $V(x) = \infty$ for $x < 0$ and $x > a$; $V(x) = 0$ for $0 < x < a$. Assume that the electrons are in the same spin state.

(a) What is the lowest energy of the two-electron state?

(b) What is the energy eigenfunction for this ground state?

(c) What is the energy and wave function of the first excited state still with the electrons having the same spin state?

(d) How do the results for (a) and (b) change if we allow the electrons to have opposite spins?

(e) How do the results for (a) and (b) change if the particles are bosons not fermions?