Solution Practice Exam 3

1. **Short answer section**

(a) What are the commutators of L_x, L_y, L_z, and L^2?

$$[L_x, L_y] = i\hbar L_z \quad [L_y, L_z] = i\hbar L_x \quad [L_z, L_x] = i\hbar L_y$$

(b) Define L_+ and L_-. Express L_x and L_y in terms of L_+ and L_-.

$$L_\pm = L_x \pm i L_y \quad L_x = \frac{1}{2}(L_+ + L_-) \quad L_y = -\frac{i}{2}(L_+ - L_-)$$

(c) What are the eigenvalues of the J^2 and J_z operators?

J^2: $\hbar^2 j(j+1)$ for $j = 0, \frac{1}{2}, 1, \cdots$ $\quad J_z$: $\hbar m$ for $m = -j, -j+1, \cdots, j$

(d) What is J_+ acting on $|j, m\rangle$? What is J_- acting on $|j, m\rangle$? What is J_z acting on $|j, m\rangle$?

$$J_\pm |j, m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\, |j, m\pm 1\rangle \quad J_z |j, m\rangle = \hbar m\, |j, m\rangle$$

(e) What are the allowed values of L^2 and L_z for the spherical harmonics? Why?

L^2: $\hbar^2 \ell(\ell+1)$ for $\ell = 0, 1, \cdots$ $\quad L_z$: $\hbar m$ for $m = -\ell, -\ell+1, \cdots, \ell$

because ψ must be single valued.

(f) What is the orthogonality condition for the spherical harmonics?

$$\langle \ell, m | \ell', m' \rangle = \delta_{\ell\ell'}\delta_{mm'} = \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi\, Y_{\ell m}(\theta, \varphi)^* Y_{\ell' m'}(\theta, \varphi)$$

(g) What is the completeness condition for the spherical harmonics?

Any function $f(\theta, \varphi)$ can be expressed as $\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell m} Y_{\ell m}(\theta, \varphi)$

(h) What is the form of the solution to the Schrodinger equation for a central potential?

$$\psi(r, \theta, \varphi) = R(r) Y_{\ell m}(\theta, \varphi)$$

(i) What is the radial Schrodinger equation for $R(r)$? What is the normalization condition for $R(r)$?

$$\left(\frac{-\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2} r + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} + V(r)\right) R(r) = E R(r) \; ; \; \int_0^\infty dr\, r^2 |R(r)|^2 = 1$$

(j) What is the radial Schrodinger equation for $u(r)$? How is $u(r)$ related to $R(r)$? What is the normalization condition for $u(r)$?

$$\left(\frac{-\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} + V(r)\right) u(r) = E u(r) \quad \int_0^\infty dr\, |u(r)|^2 = 1 \quad R(r) = \frac{u(r)}{r}$$

(k) What is the behavior of $u(r)$ as $r \to 0$ for angular momentum l and a not too singular potential?

$$u(r) \sim r^{\ell+1} \text{ as } r \to 0$$

(l) What are the solutions to the radial Schrodinger equation for a free particle ($V = 0$)? Give explicit expressions for these solutions for $l = 0$.

$R(r) = j_\ell(kr)$ and $n_\ell(kr)$, where $k = \sqrt{2mE/\hbar^2}$.

$j_0(kr) = \dfrac{\sin(kr)}{kr}$ and $n_0(kr) = -\dfrac{\cos(kr)}{kr}$

(m) What is the Bohr radius (numerically)?

$a_0 = 0.52 A° = 0.52 \times 10^{-10} m$

(n) What is a Rydberg (numerically)?

$E_I = 13.6 eV$

(o) What is the asymptotic behavior of the eigenstates of the hydrogen atom for energies less than zero?

$u(r) \sim e^{-\lambda r / a_0}$, where $\lambda = 1/n$

(p) What are the bound state energies of the hydrogen atom?

$E_n = -E_I / n^2$

(q) What are the degeneracies of the bound states of the hydrogen atom?

The degeneracy of E_n is n^2.

2. (a) $[L_+, L_z] = [L_x, L_z] + i[L_y, L_z]$
$$= -i\hbar L_y + i \cdot i\hbar L_x$$
$$= -\hbar(L_x + iL_y) = -\hbar L_+$$

$$\rightarrow [[L_+, L_z], L_-] = -\hbar[L_+, L_-]$$
$$= -\hbar[L_x + iL_y, L_x - iL_y]$$
$$= -\hbar(-i[L_x, L_y] + i[L_y, L_x])$$
$$= -\hbar(-2i)i\hbar L_z = -2\hbar^2 L_z$$

$$[[L_+, L_-], L_z] = [2\hbar L_z, L_z] = 0$$

The two commutators are not equal.

(b) This problem may be done in the same manner as the one in homework 11. However, because the function is so simple we can just write it as the Y_ℓ^m:

$$f(\theta, \varphi) = 1 + \sin\theta \sin(\varphi + \tfrac{\pi}{4})$$
$$= 1 + \sin\theta \, \frac{e^{i(\varphi + \frac{\pi}{4})} - e^{-i(\varphi + \frac{\pi}{4})}}{2i}$$
$$= 1 + \frac{e^{i\pi/4}}{2i} \sin\theta e^{i\varphi} + \frac{-e^{-i\frac{\pi}{4}}}{2i} \sin\theta e^{-i\varphi}$$
$$= 1 + \frac{e^{-i\pi/4}}{2} \sin\theta e^{i\varphi} + \frac{e^{i\pi/4}}{2} \sin\theta e^{-i\varphi}$$
$$= \sqrt{4\pi}\, Y_0^0 + \left(-\sqrt{\frac{8\pi}{3}}\right)\frac{e^{-i\pi/4}}{2} Y_1^1 + \left(\sqrt{\frac{8\pi}{3}}\right)\frac{e^{i\pi/4}}{2} Y_1^{-1}$$
$$= \sqrt{4\pi}\, Y_0^0 - \sqrt{\frac{2\pi}{3}} e^{-i\pi/4} Y_1^1 + \sqrt{\frac{2\pi}{3}} e^{i\pi/4} Y_1^{-1}$$

(c) $L_z L_+ |\ell, m\rangle = L_z \hbar\sqrt{\ell(\ell+1) - m(m+1)}\ |\ell, m+1\rangle$

$$= \hbar^2 (m+1)\sqrt{\ell(\ell+1) - m(m+1)}\ |\ell, m+1\rangle$$

For $\ell = 1$, $L_z L_+ |1,1\rangle = 0$

$L_z L_+ |1, 0\rangle = \hbar^2 \sqrt{2}\ |1,1\rangle$

$L_z L_+ |1 -1\rangle = 0$ since $m + 1 = 0$

Thus, there is only One nonzero matrix element.

$$L_z L_+ = \hbar^2 \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3. (a) $\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} + \frac{1}{2} m\omega^2 r^2 \right) u(r) = E u(r)$

(b) At large r this becomes

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{1}{2} m\omega^2 r^2 \right) u(r) \approx E u(r).$$

$$\rightarrow \quad \left(-\frac{d^2}{dr^2} + \left(\frac{m\omega}{\hbar}\right)^2 r^2 \right) u(r) \approx \frac{2mE}{\hbar^2} u(r)$$

$$\rightarrow \quad \frac{d^2 u(r)}{dr^2} \approx \left(\frac{m\omega}{\hbar}\right)^2 r^2 u(r)$$

since $\frac{2mE}{\hbar^2} \ll \left(\frac{m\omega}{\hbar}\right)^2 r^2$ for large r.

Following the hint try $e^{f(r)}$ for $u(r)$

$$\frac{de^{f(r)}}{dr} = f'(r)\, e^{f(r)}$$

$$\frac{d^2 e^{f(r)}}{dr^2} = f''(r)\, e^{f(r)} + (f'(r))^2 e^{f(r)} .$$

If $f(r) = -\alpha r^2$, then $f'(r) = -2\alpha r$ and $(f'(r))^2 = 4\alpha^2 r^2$. $f''(r) = -2\alpha$. Since $|f''(r)| \ll (f'(r))^2$ at large r,

$$\frac{d^2 e^{-\alpha r^2}}{dr^2} \approx 4\alpha^2 r^2 e^{-\alpha r^2} \rightarrow \boxed{u(r) \sim \exp\left(-\frac{m\omega}{2\hbar} r^2\right) \text{ as } r \to \infty}$$

Note that $\exp\left(+\frac{m\omega}{2\hbar} r^2\right)$ is also a solution at large r, but it is not physical since it diverges as $r \to \infty$.

(c) $u(r) \sim r^{\ell+1}$ as $r \to 0$

(d) Let $u(r) = Cr \exp\left(-\frac{m\omega}{2\hbar} r^2\right)$

$$\frac{du}{dr} = \left(-\frac{m\omega}{\hbar} r^2 + 1\right) C \exp\left(\frac{-m\omega}{2\hbar} r^2\right)$$

$$\frac{d^2u}{dr^2} = \left(-3\frac{m\omega}{\hbar} + \left(\frac{m\omega}{\hbar}\right)^2 r^3\right) C \exp\left(\frac{-m\omega}{2\hbar} r^2\right)$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = \left(\frac{3\hbar\omega}{2} r - \frac{1}{2} m\omega^2 r^3\right) C \exp\left(\frac{-m\omega}{2\hbar} r^2\right)$$

$$= \left(\frac{3\hbar\omega}{2} - \frac{1}{2} m\omega^2 r^2\right) u(r)$$

This is a solution to the differential equation for $l=0$ provided that $E=\frac{3}{2}\hbar\omega$.

(e) $H = \frac{-\hbar^2\nabla^2}{2m} + \frac{1}{2}m\omega^2 r^2$

$$= \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2\right)$$

$$+ \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2}m\omega^2 y^2\right)$$

$$+ \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{1}{2}m\omega^2 z^2\right)$$

Thus, H is the sum of onedimensional hamiltonians in the $x, y,$ and z directions, $H = H_x + H_y + H_z$. If $H_x X = E_x X$

$$H_y Y = E_y Y$$

$$H_z Z = E_z Z, \text{ then}$$

$$\begin{aligned} HXYZ &= (H_x X)YZ + X(H_y Y)Z + XY(H_z Z) \\ &= E_x XYZ + E_y XYZ + E_z XYZ \\ &= (E_x + E_y + E_z)XYZ. \end{aligned}$$

$\rightarrow E = E_x + E_y + E_z$. Since the one dimensional harmonic oscillator ground state is non-degenerate with energy $\hbar\omega/2$, the lowest $E = E_x + E_y + E_z$ can be is $3\hbar\omega/2$, and this state is non-degenerate.