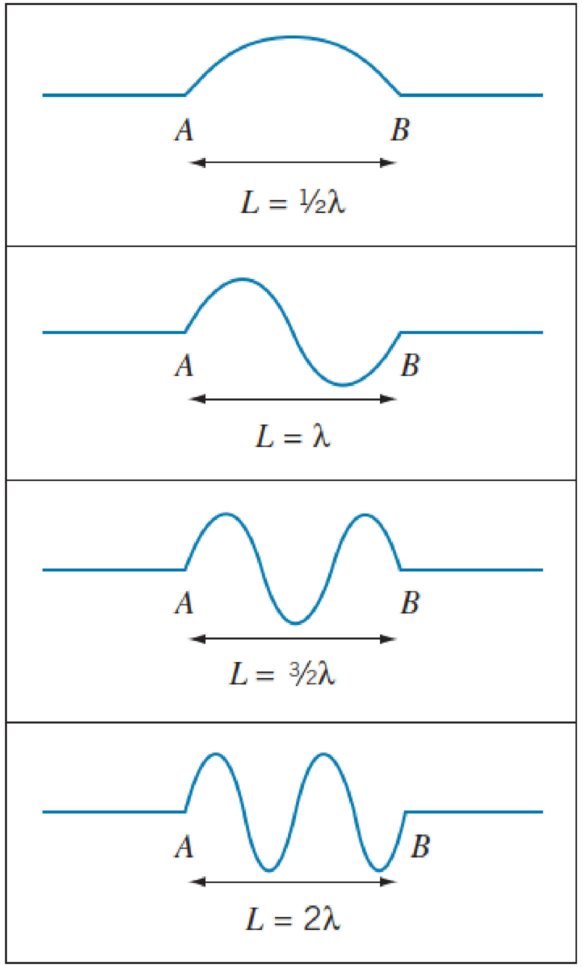
Announcements

- Homework 5 is due next Wednesday, March 1.
- Pick up your exam 1 before Friday.
- Peer teaching evaluation.

Last time



Energy quantization accompanies every attempt to confine a particle to a finite region of space.

Today’s class

- Schrödinger Equation

in-class quiz (3 min)

The lowest energy of a particle in an infinite one-dimensional well is 5.6 eV.
If the width of the well is doubled, what is its lowest energy?

a. 5.6 eV

b. 11.2 eV

c. 1.4 eV

d. 2.8 eV

e. 22.4 eV

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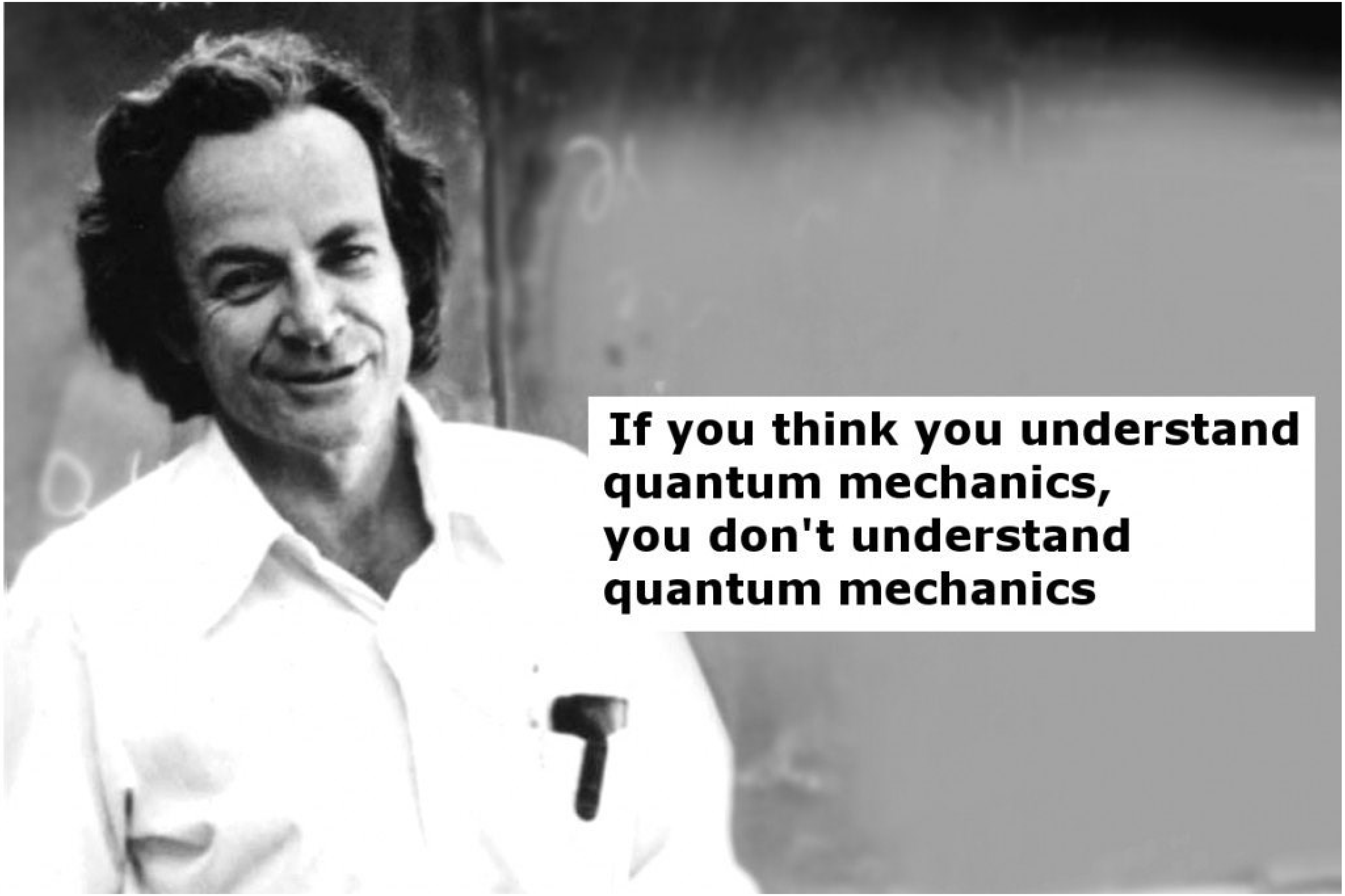
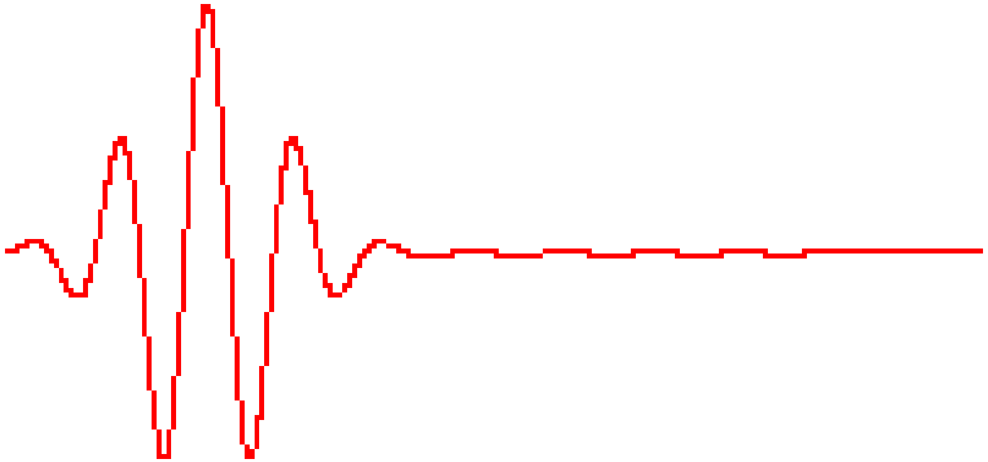
$E_n = n^2 \cdot \frac{h^2}{8mL^2}$, $E_1 = \frac{h^2}{8mL^2}$

$L' = 2L \Rightarrow E_1' = \frac{h^2}{8m(2L)^2} = \frac{1}{4}E_1$


If you think you understand
quantum mechanics,
you don't understand
quantum mechanics

Describing a particle



- What we have learnt:

- Particle has wave-like properties and is best described by a wave packet.

  ←Cannot be regarded as a 'point' because of uncertainty principle

  ←Cannot be regarded as a plane wave, not extended over infinity

- To properly describe the information regarding a particle wave packet,

  we use a **wave function, $\Psi(x,t)$**.

Schrödinger Equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi(x) = E\psi(x)$$

- The equation is the foundation of Quantum Mechanics. It cannot be derived from any previous laws or postulates.
- Wave function ψ(x) is the solution of Schrödinger Equation.
- Wave function ψ(x) describes the wave behavior of a particle.

Photo from the Nobel Foundation archive.

Erwin Schrödinger

In-class exercise (3 min)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+U(x)\psi(x)=E\psi(x)$$

Can you justify the form of Schrödinger Equation?

Hint: de Broglie wave of a free particle $\psi(x)=A\sin kx$

What's the classical relationship between the kinetic energy and momentum

$$\frac{d\psi}{dx}=Ak\cos kx,\quad \frac{d^2\psi}{dx^2}=-k^2A\sin kx$$

kinetic energy $K=\frac{p^2}{2m}=\frac{(h/\lambda)^2}{2m}=\frac{\hbar^2k^2}{2m} \Rightarrow k^2=\frac{2mK}{\hbar^2}$

$$\frac{d^2\psi}{dx^2}=-k^2\psi(x)=-\frac{2mK}{\hbar^2}\psi(x)=-\frac{2m}{\hbar^2}(E-U)\psi(x)$$

Schrödinger Equation

Time-independent Schrödinger Equation

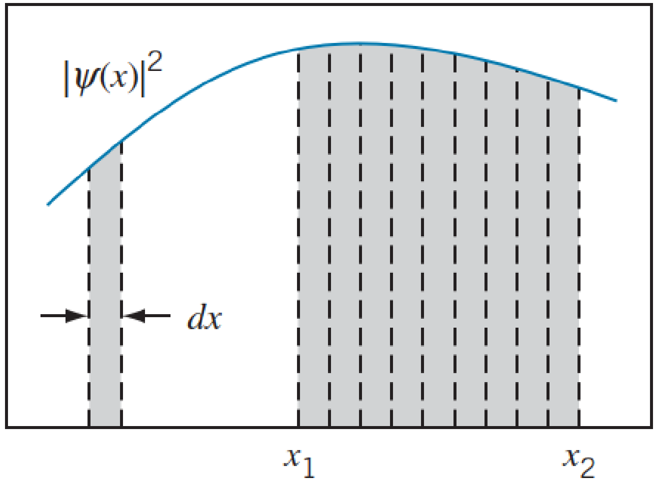
$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi(x) = E\psi(x)$$

Time-dependent Schrödinger Equation

$$\Psi(x,t) = \psi(x)e^{-i\omega t}$$ where $$\omega = E/\hbar.$$

Physical meaning of the wave function ψ(x)?

$|\psi(x)|^2$ gives the probability for finding the particle in a given region of space.



Probability density = probability per unit length

$$P(x)\,dx = |\psi(x)|^2\,dx$$

Area under the curve = probability

$$\int_{x_1}^{x_2} P(x)\,dx = \int_{x_1}^{x_2} |\psi(x)|^2\,dx$$

$E = E_0 \sin(kx - wt)$, $I = |E_0|^2$ intensity of the wave

Recall in electromagnetic waves -
What is the physical meaning of the squared absolute amplitude?

Probability density $|\psi(x)|^2$

1. Probability density is independent of time

$$|\Psi(x,t)|^2 = |\psi(x)|^2 |e^{-i\omega t}|^2 = |\psi(x)|^2$$

for time-dependent wave function $\Longrightarrow$ stationary states

Consistent with the continuity condition.

2. Total probability

$$\int_{-\infty}^{+\infty} |\psi(x)|^2\, dx = 1$$

Normalization condition

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi(x) = E\psi(x)$$

Is linear. So Cψ(x) is also a solution.

Need normalization condition to determine C.

Recipes for solving the Schrödinger Equation

Step 1: use appropriate U(x)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi(x) = E\psi(x)$$

Step 2: find a mathematical function ψ(x) that is a solution to Schrödinger Equation.

Step 3: apply boundary conditions.

Step 4: apply normalization conditions.

Once solved, we can calculate expectation values, confirm uncertainty principle, etc.

Expectation value

average outcome of a large number of measurements

$$x_{av} = \frac{\int_{-\infty}^{+\infty} P(x) x \, dx}{\int_{-\infty}^{+\infty} P(x) \, dx} = \int_{-\infty}^{+\infty} |\psi(x)|^2 x \, dx$$

the average value of any function of x

$$[f(x)]_{av} = \int_{-\infty}^{+\infty} P(x) f(x) \, dx = \int_{-\infty}^{+\infty} |\psi(x)|^2 f(x) \, dx$$

x^2, x^3, 5^x, ...

Operator

Mathematically, an operator is a function/ an operation acting on a wave function.

Physically, an operator is a measurement of an observable.

Expectation value:
Average value of measurements of a physical quantity O in state Ψ

$$\langle O \rangle = \overline{O} \equiv \int_{-\infty}^{\infty} \Psi^*(x) O \Psi(x) dx$$

Ψ* is Ψ conjugate.

Write as $\Psi^* O \Psi$ because O is an operator that acting to the right.

Example: measure average position given the wave function.

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^*(x)\, x\, \Psi(x) dx$$

Operator

Momentum operator: $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$, the expectation of the momentum:

$$\langle \hat{p}_x \rangle = \int_{-\infty}^{+\infty} \Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi(x,\mathrm{t})\,dx$$

Energy operator: $\hat{E} = i\hbar\frac{\partial}{\partial t}$, the expectation of the energy:

$$\langle \hat{E} \rangle = \int_{-\infty}^{+\infty} \Psi^*(x,t)\left(i\hbar\frac{\partial}{\partial t}\right)\Psi(x,\mathrm{t})\,dx$$

Outline of the cases we will study

- Constant potential energy
- The free particle
- Particle-in-a-box (infinite potential well)
- Particle-in-a-box (finite potential well)
- Simple Harmonic oscillator