Announcement

- Homework 5 is due on Wednesday, March 1.
- Homework 6 is due next Wednesday, March 8.

Last time

- Wave function for constant potential energy
- Free particle
- 1-D infinite potential well

$$E_n = \frac{h^2 n^2}{8mL^2} \qquad n = 1, 2, 3, \ldots$$

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \qquad n = 1, 2, 3, \ldots$$

Today's class

- Applications – finding expectation values

in-class quiz (3 min)

An electron is confined, by infinite potential energy walls on both sides, to a region of length L. If the ground state has energy 3 eV, what is the energy of the 2nd excited state?

A. 6.0 eV

B. 36.0 eV

C. 9.0 eV

D. 12.0 eV

E. 27.0 eV

in-class quiz (3 min)

An electron is confined, by infinite potential energy walls on both sides, to a region of length L. If the ground state has energy 3 eV, what is the energy of the 2nd excited state?

$n=1$ $n=3$

A. 6.0 eV

B. 36.0 eV

C. 9.0 eV

D. 12.0 eV

E. 27.0 eV

$E_n = n^2 E_0 = n^2 E_1$

$E_0 = E_1 = 3\,eV$

$E_3 = 27\,eV$

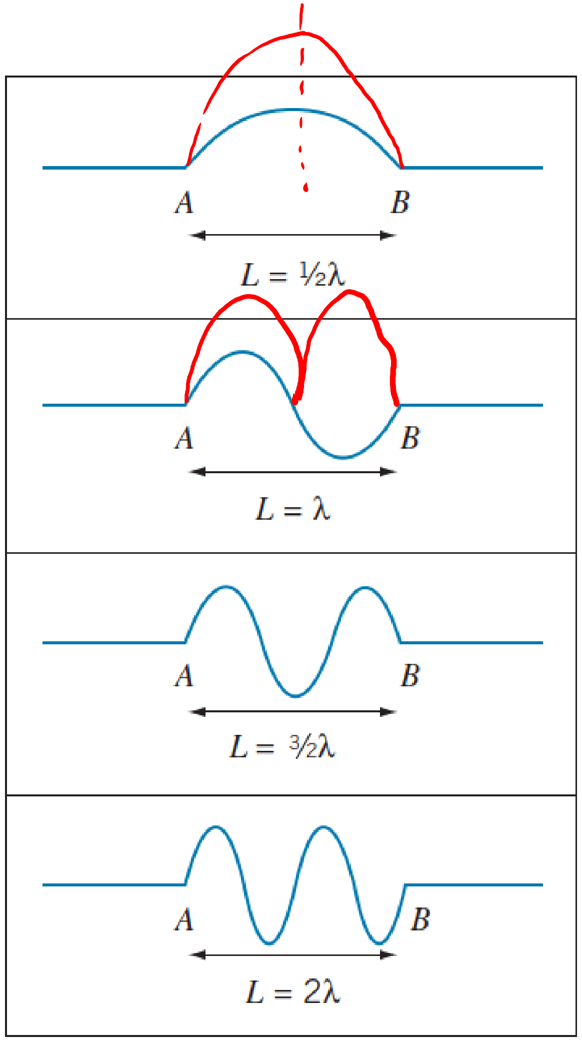
Expectation values

$<x> = \frac{L}{2}$

Can you guess?

$\psi(x)$

probability density $|\psi(x)|^2$



L/2

L/2

$$\langle \overset{x}{f(x)} \rangle = \int_{-\infty}^{\infty} \underbrace{|\psi(x)|^2}_{\text{weight}} \overset{x}{f(x)}\, dx$$

Expectation values

$$\psi(x) = \begin{cases} \sqrt{\dfrac{2}{L}} \sin\left(\dfrac{n\pi}{L} x\right) & 0 \le x \le L \\ 0 & \text{Otherwise} \end{cases}$$

$$\langle x \rangle = \int_{-\infty}^{\infty} |\psi(x)|^2 x\, dx$$

$$= \int_0^L x \cdot \frac{2}{L} \sin^2\left(\frac{n\pi}{L} x\right) dx$$

set $\frac{n\pi}{L} = a$

$$\int x \sin^2(ax) dx = \frac{x^2}{4} - \frac{x}{4a} \sin(2ax) - \frac{1}{8a^2} \cos(2ax)$$

$$= \frac{2}{L} \int_0^L x \sin^2(ax) dx = \frac{2}{L}\left[\frac{x^2}{4} - \frac{x}{4a} \sin(2ax) - \frac{1}{8a^2} \cos(2ax)\right]\Big|_0^L$$

$$\sin(2ax) = \sin\left(2 \cdot \frac{n\pi}{L} x\right)\Big|_0^L = \sin\left(2 \cdot \frac{n\pi}{L} \cdot L\right) - \sin\left(2 \cdot \frac{n\pi}{L} \cdot 0\right) = \sin(2n\pi) - \sin 0 = 0 - 0 = 0$$

$$\cos(2ax) = \cos\left(2 \cdot \frac{n\pi}{L} x\right)\Big|_0^L = \cos\left(2 \cdot \frac{n\pi}{L} \cdot L\right) - \cos\left(2 \cdot \frac{n\pi}{L} \cdot 0\right) = \cos(2n\pi) - \cos 0 = 1 - 1 = 0$$

$$= \frac{2}{L} \frac{x^2}{4}\Big|_0^L = \frac{2}{L}\left(\frac{L^2}{4} - \frac{0^2}{4}\right) = \frac{L}{2}$$

In-class exercise (3 min)

$$\psi^* = \psi(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & 0 \le x \le L \\ 0 & \text{otherwise} \end{cases}$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx$$

$$= \int_0^L x^2 \cdot \frac{2}{L} \sin^2\left(\frac{n\pi}{L} x\right) dx$$

$$= \frac{2}{L} \int_0^L x^2 \sin^2\left(\frac{n\pi}{L} x\right) dx, \quad \text{set } \frac{n\pi}{L} = a$$

$$= \frac{2}{L} \int_0^L x^2 \sin^2 ax \, dx = \frac{2}{L} \Big[\qquad \Big]\Big|_0^L$$

$$\sin(2ax) = \cdots\cdots = 0 - 0 = 0$$

$$\cos(2ax) = \cdots\cdots = 1 - 1 = 0$$

$$= \frac{2}{L}\left(\frac{L^3}{6} - \frac{L^3}{4n^2\pi^2}\right) = L^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right)$$

$$\int x^2 \sin^2(ax)dx = \frac{x^3}{6} - \left[\left(x^2 - \frac{1}{2a^2}\right)\sin(2ax)/4a\right] - \frac{x}{4a^2}\cos(2ax)$$

$$\frac{x}{4a^2}\cos(2ax) = \left[\frac{x}{4\frac{n^2\pi^2}{L^2}} \cos\left(2 \cdot \frac{n\pi}{L} \cdot x\right)\right]\Big|_0^L$$

$$= \frac{L}{4\frac{n^2\pi^2}{L^2}} \cdot \cos(2n\pi) - \frac{0}{4\frac{n^2\pi^2}{L^2}} \cos 0$$

$$= \frac{L^3}{4n^2\pi^2}$$

Expectation values

$\psi^* = \psi = \sqrt{\frac{2}{L}} \sin(\frac{n\pi x}{L})$ $\quad 0 \le x \le L$

$$\hat{p}_x = -i\hbar \frac{\partial}{\partial x}$$

$<p> = \int \psi^* \hat{p} \psi \, dx$

<p> = 0 Your educated guess?

$<p> = \int_{-\infty}^{\infty} \psi^* (-i\hbar \frac{\partial}{\partial x}) \psi \, dx = \int_0^L \sqrt{\frac{2}{L}} \sin(\frac{n\pi x}{L}) (-i\hbar \frac{\partial}{\partial x}) \sqrt{\frac{2}{L}} \sin(\frac{n\pi x}{L}) \, dx$

$= \frac{2}{L}(-i\hbar) \int_0^L \sin(\frac{n\pi x}{L}) \underbrace{\frac{\partial}{\partial x} \sin(\frac{n\pi x}{L})} \, dx$

$\frac{\partial}{\partial x} \sin(\frac{n\pi x}{L}) = \frac{n\pi}{L} \cos(\frac{n\pi x}{L})$

$= \frac{2}{L}(-i\hbar) \cdot \frac{n\pi}{L} \underbrace{\int_0^L \sin(\frac{n\pi x}{L}) \cos(\frac{n\pi x}{L}) \, dx}$

$\sin 2\alpha = 2 \sin\alpha \cos\alpha$

$\int_0^L \frac{1}{2} \sin(2 \cdot \frac{n\pi x}{L}) \, dx = \int_0^L \underbrace{\frac{1}{2}(-1) \cdot \frac{L}{2n\pi}} \, d\cos(\frac{2n\pi}{L} x) = -\frac{L}{4n\pi} \cos(\frac{2n\pi}{L} x) \Big|_0^L$

$= -\frac{L}{4n\pi} (\cos 2n\pi - \cos 0) = -\frac{L}{4n\pi} (1-1) = 0$

$<p> = 0$

$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$, $\hat{p}_x^2 = \left(-i\hbar\frac{\partial}{\partial x}\right)^2 = -\hbar^2\frac{\partial^2}{\partial x^2}$ $\qquad \psi^* = \psi(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \quad 0 \le x \le L$

Expectation values

$i^2 = -1$

$(-1)^2 = 1$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \psi^* \hat{p}_x^2 \psi \, dx = \int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax)$$

$$= \frac{2}{L}(-\hbar^2)\int_0^L \sin\left(\frac{n\pi x}{L}\right)\frac{\partial^2}{\partial x^2}\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\frac{\partial^2}{\partial x^2}\sin\left(\frac{n\pi x}{L}\right) = \frac{\partial}{\partial x}\cdot\frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right) = -\left(\frac{n\pi}{L}\right)^2\sin\left(\frac{n\pi x}{L}\right)$$

$$= \frac{2}{L}(-\hbar^2)\left(-\frac{n^2\pi^2}{L^2}\right)\underbrace{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx}$$

set $\frac{n\pi}{L} = a$

$$\int_0^L \sin^2(ax)\,dx = \left[\frac{x}{2} - \frac{1}{4a}\sin(2ax)\right]\Bigg|_0^L = \frac{L-0}{2} - \left[\frac{1}{4a}\sin(2a\cdot L) - \frac{1}{4a}\sin(2a\cdot 0)\right]$$

$$= \frac{L}{2} - \left[\frac{1}{4a}\cdot\cancelto{0}{\sin\left(2\cdot\frac{n\pi}{L}\cdot L\right)} - \frac{1}{4a}\cancelto{0}{\sin 0}\right]$$

$$= \frac{L}{2}$$

$$= \frac{2}{L}\hbar^2\cdot\frac{n^2\pi^2}{L^2}\cdot\frac{L}{2}$$

$$= \frac{\hbar^2\pi^2\cdot n^2}{L^2}$$