Phase Effects in the Diffraction of Light: Beyond the Grating Equation

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Diffraction gratings affect the absolute phase of light in a way that is not obvious from the usual derivation of optical paths using the grating equation. For example, consider light which encounters first one and then the second of two parallel gratings. If one grating is moved parallel to its surface, the phase of the light diffracted from the grating pair is shifted by 2π each time the grating is moved by one grating constant, even though the geometric path length is not altered by the motion. This additional phase shift must be included when incorporating diffraction gratings in interferometers.

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Our understanding of diffraction gratings relies on the grating equation [1,2], which says that if light of wavelength λ is incident on a grating with grating constant d at an angle α (relative to the grating normal), then the diffracted light leaves the grating at a wavelength-dependent angle β satisfying $d(\sin\alpha + \sin\beta) + m\lambda = 0$, with m an integer indicating the diffraction order [3]. One obtains this result by requiring that the phase of the diffracted light from adjacent grating rulings or slits differs by $2\pi m$ (which has the same effect as zero difference) so that the diffracted waves from all slits interfere constructively in the direction of the diffracted beam.

Given this understanding, one can imagine constructing a gadget like the one shown in Fig. 1, where a pair of parallel and identical face-to-face gratings is used to diffract light twice, so that the outgoing waves are parallel to, but laterally displaced from, the incoming waves. According to the grating equation, the longer wavelength light is diffracted through a larger angle than the shorter wavelength light, so that the red light strikes the second grating to the left of the green while the blue light is to the right of the green. One can see by inspection that the total free-space path taken by a red ray is longer than that of a green ray, which is in turn longer than the blue ray's path. (In short-pulse laser applications, such arrangements of parallel gratings are often used in pulse compressors [4].)

Finally, then, if one puts mirrors at the places shown in the figure, and adjusts the lengths of the common paths and the intergrating spacing D correctly, it should be possible to arrange for each color that the ratio of free-space path to the wavelength is the *same* integer value. If this were so, and if the gratings had no other effect on the phase of the light waves, then the device shown would be a cavity resonant for all wavelengths, a ''white-light'' cavity. Detailed calculations [5] show that the bandwidth of this cavity would in fact be finite (because of the nonlinear dispersion of the gratings) but would be many orders of magnitude larger than the bandwidth of the typical Fabry-Perot cavity, such as the ones in the arms of the LIGO detector [6,7].

In this Letter we describe measurements of the phase shift of light by such a parallel grating set which show that the above concept is almost *completely wrong*. Instead, the pair of gratings provides a wavelength-dependent phase shift nearly canceling the phase from the additional free-space path length shown in Fig. 1 [8]. We proved this to ourselves by measuring the resonance bandwidth of Fabry-Perot cavities containing high-efficiency gratings and configured to be ''white-light'' cavities. We found no enhancement of bandwidth. Here we show why the expected

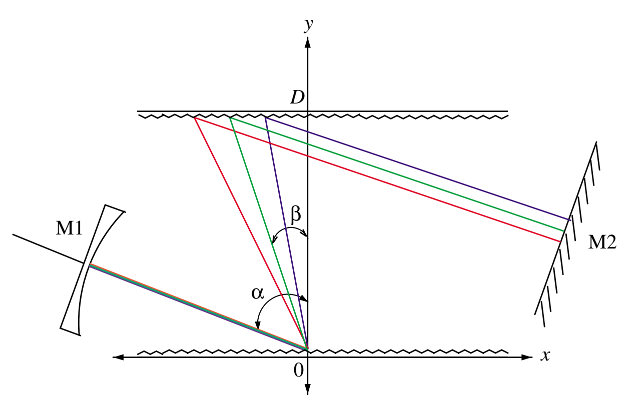

FIG. 1 (color). A pair of identical, parallel, and face-to-face diffraction gratings and two mirrors (M1 and M2) form a resonant cavity. The light rays approaching and leaving the grating pair are parallel for all wavelengths but take different paths after the first grating. (From left to right the rays are red, green, and blue.)

enhancement does not occur. Moreover, we also show that the phase depends not only on the intergrating spacing, but also on the exact relationship between the grating features as seen by the light. This dependence leads to the non-intuitive result that the phase is modulated strongly if one of the gratings is translated *parallel to its face*, even though the optical paths in Fig. 1 are wholly unaffected.

The error lies with an inappropriate mix of geometrical and physical optics. Consider the effect of the parallel grating pair on infinite plane waves. In Fig. 1, parallel reflective gratings are located in the $y = 0$ and $y = D$ planes. We will calculate the electric field at the two gratings and in a plane normal to the outgoing light (where the right-hand mirror in Fig. 1 is located). The light field impinging from the left on the first grating is

$$E_{1,\mathrm{in}} = E_0 e^{ik(x\sin\alpha - y\cos\alpha)}. \tag{1}$$

The grating at $y = 0$ bestows a spatial phase modulation on the incoming plane wave. The phase factor is $e^{ikG(x-x_0')}$; the periodic function $G(x - x_0')$ represents the grating profile, with origin at x_0'. This phase factor may be expanded in a Fourier series:

$$e^{ikG(x-x_0')} = \sum_m C_m e^{img(x-x_0')}, \tag{2}$$

where $g = 2\pi/d$.

Each term of the series is a diffraction order. In the following we consider only the $m = -1$ order, set $C_{-1} = 1$, and chose $x_0' = 0$ for the first grating. The light field leaving that grating is

$$E_{1,\mathrm{out}} = E_0 e^{i[(k\sin\alpha - g)x + ky\cos\beta]}, \tag{3}$$

where we have used the grating equation to substitute $k\sin\alpha + mg$ for $-k\sin\beta < 0$. When the light reaches the second grating, on the $y = D$ line, it again receives a spatial phase modulation $e^{img(x_0-x)}$. The quantity x_0 is the x offset of the second grating's periodic modulation with respect to that of the first grating. Note that the second grating is reversed relative to the first, so that its local coordinate runs in the $-x$ direction. We again use $m = -1$, making light of all wavelengths leave the second grating parallel to the incident light. The outgoing electric field at the second grating is

$$E_{2,\mathrm{out}} = E_0 e^{i[k(x\sin\alpha + D\cos\beta) - gx_0]}. \tag{4}$$

When the light finally arrives at a point (x, y) on the right-hand mirror of Fig. 1, the electric field will be

$$E_{\mathrm{em}} = E_0 e^{i[k\{x\sin\alpha - (y-D)\cos\alpha + D\cos\beta\} - gx_0]}. \tag{5}$$

We now consider the phase, $\Phi(\omega, x, y) = (\omega/c) \times [x\sin\alpha - (y - D)\cos\alpha + D\cos\beta] - gx_0$, at M2. This may be written as

$$\Phi(\omega) = \frac{\omega}{c} L(\omega) - g(D\tan\beta + x_0), \tag{6}$$

where

$$L(\omega) = x\sin\alpha - (y - D)\cos\alpha + D\left(\frac{1}{\cos\beta} + \tan\beta\sin\alpha\right) \tag{7}$$

is the total, frequency-dependent, geometric path from the first grating (at the origin) to the end mirror, as sketched in Fig. 1. We next compute the dispersion $\partial\Phi/\partial\omega$, using the grating equation to eliminate $\partial\beta/\partial\omega$, and find

$$\frac{\partial\Phi}{\partial\omega} = \frac{L(\omega)}{c}. \tag{8}$$

Equation (8) makes it clear that the variation of phase with frequency cannot be set to zero. Earlier calculations by some of us [5] used only the geometric path length contribution to the phase, $\omega L(\omega)/c$—the first term in Eq. (6)—to predict (incorrectly) that $\partial\Phi/\partial\omega$ could become zero, thus allowing for the possibility of a white-light cavity. Missing from Ref. [5] was the second term in the right-hand side of Eq. (6), present due to the position-dependent phase shift that light receives upon reflection from (or transmission through) the grating pair.

The result of Eq. (8) is familiar to short-pulse laser physicists as the group delay [9]. The exact form of the additional phase shift is rarely a concern, as the pulse compressor does not depend upon absolute phases. To our knowledge, direct experimental verification of the phase shift's form has never been published. The D and x_0 dependence of the phase in Eq. (6) may be expressed in a particularly direct way as

$$\Phi(\omega) = \frac{\omega}{c}[L_0 + D(\cos\alpha + \cos\beta)] - gx_0 \tag{9}$$

where $L_0 = x\sin\alpha - y\cos\alpha$ is the perpendicular distance from the origin to the plane of M2. Analyzing the grating compressor with plane waves reveals the origin and significance of the position-dependent phase shift on reflection from the gratings. We may now resume calculations with the geometrical optical path, so long as we do not neglect the additional phase associated with the gratings. This theory makes a very specific prediction, which may be experimentally confirmed, about how the one-way phase shift depends on the distance D between gratings and the spatial offset x_0 between grating profiles.

To test that the phase shift does have the specific form of Eq. (9), we incorporated a pair of gratings into one arm of a Mach-Zehnder interferometer, as shown in Fig. 2. We used reflective gratings with 1500 grooves/mm and a design input angle $\alpha = 42°$ (so that $\beta = 68°$) We also used an input angle of $\alpha = 50°$ ($\beta = 57.3°$)in some of our trials. The gratings had a high efficiency, with 94–96% of the incident light diffracted in first order. We placed the second grating of our grating pair on a two-axis translation stage. This stage allowed us to vary the parameters x_0 and D of the grating pair. We aligned the grating face with one of the

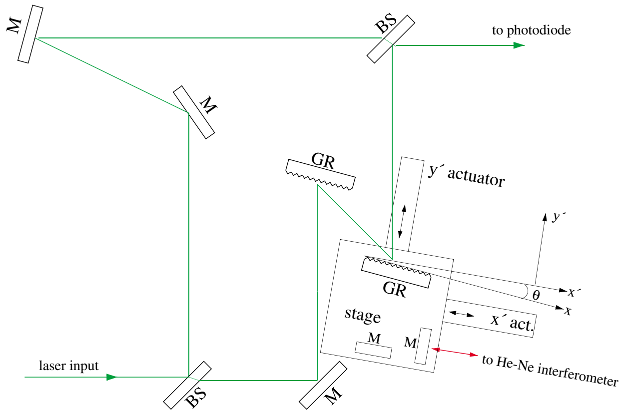

FIG. 2 (color online). The second of a pair of parallel gratings within one arm of a Mach-Zehnder interferometer is placed on an x-y translation stage. Michelson interferometers with He-Ne lasers monitor the motion of the stage.

two orthogonal axes of the stage, which we will call x' and y', as well as was possible with a naked human eye. A small misalignment angle θ inevitably remains between the grating axes (x, y) and the stage axes (x', y'). The axes are related by $\hat{x}' = \hat{x}\cos\theta + \hat{y}\sin\theta$. Assuming the construction of the translation stage is better than our ability to place the grating, we also have $\hat{y}' = -\hat{x}\sin\theta + \hat{y}\cos\theta$. As the stage moves along the x' direction, it will produce a combination of the effects on phase due to the phase's x_0 and D dependence; however, because θ is small, the influence of x_0, with a period equal to the grating period, will dominate. The converse is true when one moves the stage along y'.

To calibrate the displacement of the translation stage, we attached to it the end mirror of a simple Michelson interferometer illuminated by a helium-neon laser. In fact, there are two mirrors (and two interferometers) set perpendicular to the two motions of the stage. To ensure that perpendicularity, we move the stage in the orthogonal direction, so that the mirror slews crabwise across the He-Ne beam, and adjust its angle relative to the stage until we reduce the number of output intensity fringes to a minimum. Again, this technique relies upon good inherent perpendicularity in the stage's crossed axes. Whenever the stage moves, we monitor the output intensity of both interferometers. We quantify the stage motion by counting the He-Ne fringes.

The input to the Mach-Zehnder interferometer is a 1064 nm-wavelength grating-stabilized diode laser. For good contrast, the physical lengths of the two arms are nearly equal. We move the second grating along the x' and y' axes and observe the intensity fringes of the infrared interferometer and compare with theory. While the light input angle α and the grating period d are known, θ remains as a fitting parameter. The output intensity of the Mach-Zehnder is fit to

$$I(x') = A + B\cos[\Phi(x') + C], \tag{10}$$

where

$$\Phi(x') = \frac{2\pi}{\lambda}(\cos\alpha + \cos\beta)x'\sin\theta + \frac{2\pi}{d}x'\cos\theta, \tag{11}$$

for the x' motion, or to

$$I(y') = A + B\cos[\Phi(y') + C], \tag{12}$$

with

$$\Phi(y') = \frac{2\pi}{\lambda}(\cos\alpha + \cos\beta)y'\cos\theta - \frac{2\pi}{d}y'\sin\theta, \tag{13}$$

for the orthogonal direction. The quantities A, B, and C are rather unimportant fitting parameters; the period of the output fringes determined by Φ is key. We make a least-square fit of the theory to our data by adjusting A, B, C, and θ. Figure 3 shows examples of typical results for a trial with $\alpha = 50°$.

The top panel of Fig. 3 shows the interference seen for movement along the x'-direction, i.e., when the grating moves parallel to its face. This motion gives strong fringes; the measured fringe contrast is in the 92–94% range. Now, motion parallel to the grating face has no effect on the geometric path lengths inside the interferometer. Thus, our initial expectation (based on the geometric path length) was that the light phase would be unaffected by this motion. In contrast to this expectation, the phase of the light goes through a full cycle as the grating is translated by an amount d.

In the bottom panel of Fig. 3, we show the interference signal observed for motion along y'. We also plot, in

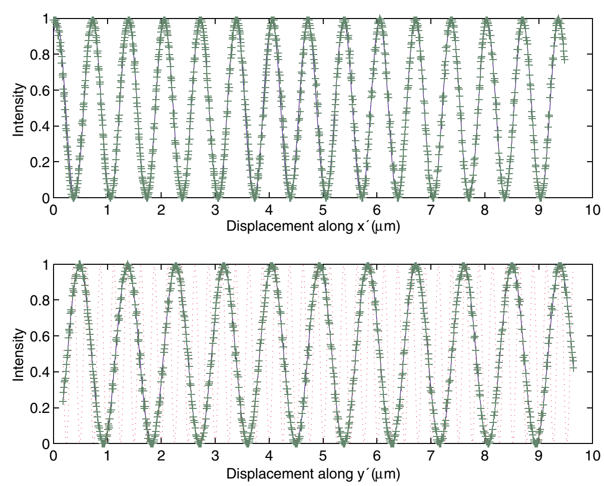

FIG. 3 (color online). (Top) Measured (crosses) and theoretical (solid line) data for motion parallel to the grating face. The light is incident at 50°. The fit to Eqs. (10) and (11) required a misalignment angle of $\theta = -0.2° \pm 0.2°$. (Bottom) The crosses show the data for motion perpendicular to the grating face. Of the two calculations, the results favor the one using Eqs. (12) and (13), with the additional phase shift on reflection (solid line) over one based on geometric path length alone (dotted line). The fit required a misalignment angle of $\theta = -0.10° \pm 0.07°$.

addition to the predicted output from the theory above, the output intensity that would be observed if only the geometric path length were changed by the grating motion. Use of geometric path alone predicts a period for the interference pattern that is different from the measured one, whereas the theory that incorporates the position-dependent phase shift predicts the period that we measured. The outcome makes clear that the setup cannot be understood with only the diffraction angle and the geometric path length. The additional position-dependent phase shift is real.

Figure 4 shows an indication of the agreement between experiment and the theory presented here. In it, we plot the misalignment angle θ of the grating for a number of trials. Eleven of the twenty-three measurements are derived from grating motion parallel to its face, and 12 from perpendicular motion. In every case, we obtained high-contrast fringes with agreement with theory comparable to what is shown in Fig. 3. The error bars on each datum reflect uncertainty in $\alpha, d, \theta, \lambda$, and the motion of the stage. Clusters of specific values for θ indicate the systematic error in the alignment of the grating on the stage, but the overall errors are very small. The quality of the fit to the measured interference pattern is evident in Fig. 3 and in the small values for the misalignment angles in Fig. 4, averaging $\theta = 0.03° \pm 0.12°$, a reasonable value for alignment by human eye.

The phase of light reflected by or transmitted through a diffraction grating cannot be deduced from the grating equation alone. That equation omits the curious result, derived above, that the absolute phase is proportional to the distance along the grating face at which the light strikes. Indeed, the flat gratings behave as mirrors tilted at angles $\Phi_{\text{tilt}} = \sin^{-1}(m\lambda/2d)$ relative to the x axis shown in Fig. 1. We confirmed this theory by testing the dependence of light phase on the position of the grating. For a grating-compressor setup, we found good agreement between this theory and the change of light phase as the mirror moved both parallel and perpendicular to its face. Our result shows that white-light cavities cannot be built from grating pairs. In fact, one might have conjectured that causality should prevent white-light cavities from being built in a much wider class of nondissipative systems—not just grating pairs. Indeed, we have found that a pair of prisms has a similar effect to the gratings on the phase of light passing through them. Finally, we note that the phase effect discussed here is not unique to the grating pair and would arise in an experiment utilizing a single grating. In our arrangement, the first grating is fixed and serves to preserve the beam width and to keep the angle of the light leaving the second grating constant as wavelength is adjusted. Otherwise, it is equivalent to a mirror. Except for a loss of contrast, we expect that the data of Fig. 3 would be identical if the first grating were replaced by an appropriately oriented mirror.

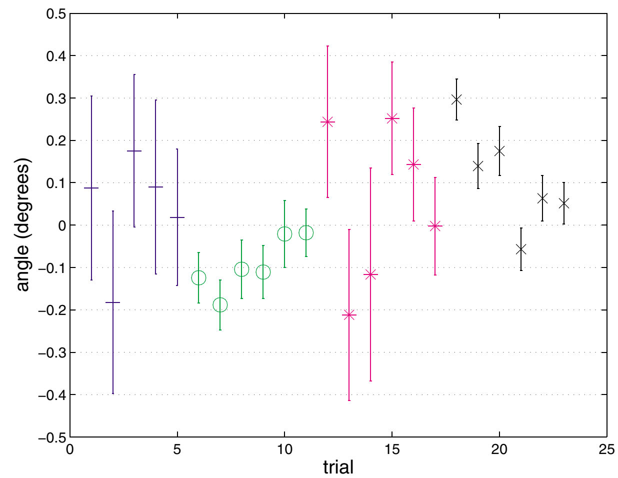

FIG. 4 (color online). Misalignment angle θ as determined from fits of measured interference data to theory. The meaning of the symbols is as follows. Crosses: motion along x' with $\alpha = 50°$; Circles: motion along y' with $\alpha = 50°$; Asterisks: motion along x' with $\alpha = 42°$; x's: motion along y' with $\alpha = 42°$. The data in Fig. 3 are from trials 2 and 8.

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