PHY 4324—-Electromagnetism 2–Spring 2021

Problem set 3—Due Wednesday, February 17th 10:00 pm

10 points each problem.

You must scan your solutions and upload to Canvas by the due time. I will upload solutions shortly afterward and homework turned late will not be accepted.

1. Read section 9.1.3 about boundary conditions for waves at an interface between two media. (The discussion is about waves on a string but apply to waves quite generally.) A wave is traveling along the x axis and that at $x = 0$ there is an interface. Changing notation sightly, let's suppose that the media are numbered 1 (left) and 2 (right). To the right, the wave velocity is v_2. To the left, the wave has velocity v_1, different from v_2.

   Both media support waves of the form

   $$f = f(x - vt),$$

   moving to the right, and

   $$f = f(x + vt),$$

   moving to the left.

   The boundary conditions are that the wave and its space derivative are continuous at the boundary. (See Eq. 9.26 and 9.27.)

   Let's now set a wave in the left medium ($x < 0$) moving towards the interface. Call it the "incident wave":

   $$f_I = f_I(x - v_1 t).$$

   We expect that there will be a wave transmitted into the right medium that looks like

   $$f_T = f_T(x - v_2 t).$$

   We should also allow that there will be a reflected wave in the left medium that looks like

   $$f_R = f_R(x + v_1 t).$$

   Use the boundary conditions to find the form that $f_T(x - v_2 t)$ and $f_R(x + v_1 t)$ must have, in terms of $f_I(x - v_1 t)$, v_1 and v_2.

2. Any function of the form $f(x - vt)$ is a wave (in one dimension) and a solution to the wave equation. This problem is about such functions as electromagnetic waves. The most general fields can be written

   $$\vec{E} = \hat{x} f_x(x - ct) + \hat{y} f_y(x - ct) + \hat{z} f_z(x - ct)$$

   $$\vec{B} = \frac{1}{c}[\hat{x} g_x(x - ct) + \hat{y} g_y(x - ct) + \hat{z} g_z(x - ct)]$$

   where the 6 functions $f_x \ldots g_z$ are 6 arbitrary functions that have finite second derivatives. They can be quite different from each other and still satisfy the wave equation.

Note that the wave is traveling in the x direction so that the spatial variation is only a function of x. (The factor $1/c$ is there to get the units right.) Show that Maxwell's microscopic equations (with no charges or currents) require that $f_x = 0$, $g_x = 0$, $f_y = g_z$ and $f_z = -g_y$. Thus although I started with 6 functions, only 2 are really arbitrary. (They correspond to two independent polarizations.)

3. Suppose waves are traveling in a region where total charges and total currents are present $(\rho, \vec{j} \neq 0)$ but without a material medium. Use Maxwell's microscopic equations to derive wave equations for $\vec{E}$ and $\vec{B}$.

   Hint: The wave equation for $\vec{E}$ will contain only $\vec{E}$, ρ, $\vec{j}$ and c. The wave equation for $\vec{B}$ will contain only $\vec{B}$, $\vec{j}$ and c. (both might have also ϵ_0 and μ_0.)

4. One usually needs the time-averaged fields in working with things like energy density and intensity. If the instantaneous energy or intensity is varying as $\cos^2(\omega t)$ we know that the time average is just half of the peak value.

   a. Now suppose that you have two cosines of the form

   $$E = \cos(kx - \omega t + \delta_e)$$

   and

   $$D = \cos(kx - \omega t + \delta_d)$$

   (They have unit amplitude.) Calculate the time average of

   $$\langle E \cdot D \rangle$$

   You may do this at $x = 0$ if you want.

   b. Now suppose the fields are

   $$E = e^{i(kx - \omega t + \delta_e)}$$

   and

   $$D = e^{i(kx - \omega t + \delta_d)}$$

   (They still have unit amplitude.) Show that calculating the following

   $$\frac{1}{2}\,\mathrm{Re}(E \cdot D^*)$$

   gives the same answer as the time average found in part a. Here, Re means "real part" and the star implies complex conjugation

   c. Show that

   $$\frac{1}{2}\,\mathrm{Re}(E^* \cdot D)$$

   is also the same.

   d. Use this idea to propose equations for time averaged energy density $\langle u \rangle$ and Poynting vector $\langle \vec{S} \rangle$.