2021 P3

1) BC: $f_{left}(x=0, t) = f_{right}(x=0, t)$ (1)

$$\left.\frac{\partial f_{Left}}{\partial x}\right|_{x=0} = \left.\frac{\partial f_{right}}{\partial t}\right|_{x=0} \quad (2)$$

(1) $\Rightarrow$ $f_I(0, t) + f_R(0, t) = f_T(0, t)$

or

$$f_I(-v_1 t) + f_R(+v_1 t) = f_T(-v_2 t) \quad (3)$$

For all 3 waves $\frac{\partial f}{\partial x} = -\frac{1}{v}\frac{\partial f}{\partial t}$. So (2) $\Rightarrow$

$\uparrow v < 0$ for f_R

$$-\frac{1}{v_1}\frac{\partial f_I}{\partial t} + \frac{1}{v_1}\frac{\partial f_R}{\partial t} = -\frac{1}{v_2}\frac{\partial f_T}{\partial t}$$

Integrate $\int dt\,[\;]$

$$-\frac{1}{v_1} f_I(-v_1 t) + \frac{1}{v_1} f_R(+v_1 t) = -\frac{1}{v_2} f_T(-v_2 t) + C, \quad C\text{: constant}$$

or multiply $(-v_1)$

$$f_I(-v_1 t) - f_R(+v_1 t) = \frac{v_1}{v_2} f_T(-v_2 t) - v_1 C$$

$$= \frac{v_1}{v_2} f_T(-v_2 t) - C' \quad (4) \quad C' = v_1 C$$

Add (3) + (4)

$$2 f_I = \left(1 + \frac{v_1}{v_2}\right) f_T - C'$$

$$f_T = \frac{1}{1 + \frac{v_1}{v_2}}\left(2 f_I + C'\right)$$

$$f_T = \frac{2 v_2}{v_1 + v_2} f_I + \frac{v_2 C'}{v_1 + v_2}$$

Subtract (3) − (4)

$$2 f_R = \left(1 - \frac{v_1}{v_2}\right) f_T + c'$$

substitute for f_T

$$2 f_R = \left(1 - \frac{v_1}{v_2}\right) \frac{2 v_2}{v_1 + v_2} f_I + \left(1 - \frac{v_1}{v_2}\right) \frac{v_2 c'}{v_1 + v_2} + c'$$

$$f_R = \frac{v_2 - v_1}{v_1 + v_2} f_I + \frac{1}{2} \frac{v_2 - v_1}{v_1 + v_2} c' + \frac{1}{2} c'$$

$$= \frac{v_2 - v_1}{v_2 + v_1} f_I + \frac{1}{2} \frac{v_2 - v_1 + v_1 + v_2}{v_1 + v_2} c'$$

$$= \frac{v_2 - v_1}{v_2 + v_1} f_I + \frac{v_2}{v_2 + v_1} c'$$

Let $c'' = C = \frac{v_2}{v_2 + v_1} c'$ if you want

2. For any function $f(x-vt)$ I'll have

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} \qquad u = x - vt \qquad \frac{\partial u}{\partial x} = 1$$

$$= \frac{\partial f}{\partial u} = f' \quad \text{also} \quad \frac{\partial f}{\partial y} = 0 = \frac{\partial f}{\partial z}$$

$$\nabla \cdot \vec{E} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\partial E_x}{\partial x} = \frac{\partial f_x}{\partial x} = 0 = \frac{\partial f}{\partial u}$$

$f_x' = 0 \Rightarrow f_x = \text{const}$ $\quad f_x'' = 0 \quad$ f_x is not a solution because $f_x \neq f(x-ct)$

$\nabla \cdot \vec{B} = 0$ same $\Rightarrow$ g_x is not a solution.

So now I have

$$\vec{E} = \hat{y} f_y + \hat{z} f_z \qquad \vec{B} = \hat{y} g_y + \hat{z} g_y$$

Moreover $\nabla \times \vec{E} = -f_z' \hat{y} + f_y' \hat{z}$ (only terms with $\frac{\partial}{\partial x}$)

$$\nabla \times \vec{B} = -\frac{1}{c} g_z' \hat{y} + \frac{1}{c} g_y' \hat{z}$$

For any $f(x-vt)$ I'll have

$$\frac{\partial f}{\partial t} = -v \frac{\partial f}{\partial u} = -v f' \quad \text{because} \quad \frac{\partial u}{\partial t} = -v$$

Thus $\nabla \times E = -\frac{\partial B}{\partial t}$ becomes

$$-\hat{y} f_z' + \hat{z} f_y' = -\frac{1}{c}(-v)(\hat{y} g_y' + \hat{z} g_z') = \frac{v}{c}(\hat{y} g_y' + \hat{z} g_z')$$

So $-f_z' = \frac{v}{c} g_y' \qquad f_y' = \frac{v}{c} g_z'$

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial E}{\partial t}$$

$$-\frac{1}{c} g_z' = -\frac{v}{c^2} f_y' \qquad +\frac{1}{c} g_y' = -\frac{v}{c^2} f_z'$$

substitute f_y'

$$-\frac{1}{c} g_z' = -\frac{v}{c^2}\frac{v}{c} g_z' \quad \text{or} \quad 1 = \frac{v^2}{c^2} \Rightarrow v = c$$

1b2 so then

$g_y{}' = -f_z{}'$ $\quad g_z{}' = f_y{}'$

Integrate

$\int dh$ C

$g_y = -f_z$ $\quad g_z = f_y$

3 Maxwell's eqns are

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Lets do $\nabla \times (\nabla \times \vec{E})$ first. $\epsilon_0 \mu_0 = \frac{1}{c^2}$

identity

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t} \nabla \times \vec{B}$$

Maxwell

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 E = -\frac{\partial}{\partial t}(\mu_0 \vec{J}) - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

but $\nabla \cdot \vec{E} = \frac{1}{\epsilon_0}\rho$

so

$$\boxed{\frac{1}{\epsilon_0} \nabla \rho + \mu_0 \frac{\partial \vec{J}}{\partial t} = \nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}}$$

similarly $\nabla \times (\nabla \times \vec{B})$ becomes

$$\nabla \times (\nabla \times \vec{B}) = \mu_0 \nabla \times \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \nabla \times \vec{E}$$

Maxwell

$$\nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = \mu_0 \nabla \times \vec{J} + \mu_0 \epsilon_0 \left(-\frac{\partial^2 \vec{B}}{\partial t^2}\right)$$

but $\nabla \cdot \vec{B} = 0$

so

$$\boxed{-\mu_0 \nabla \times \vec{J} = \nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}}$$

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4. $\langle E \cdot D \rangle = \frac{1}{T}\int_0^T dt \cos(-\omega t+\delta_e)\cos(-\omega t+\delta_d)$ $\quad x=0$

You can use identities but you can also write
$\cos\phi = \frac{1}{2}(e^{i\phi}+e^{-i\phi})$

$$\langle E \cdot D \rangle = \frac{1}{T}\frac{1}{4}\int_0^T dt\,\left(e^{i(\delta_e-\omega t)}+e^{i(\omega t-\delta_e)}\right)\left(e^{i(\delta_d-\omega t)}+e^{i(\omega t-\delta_d)}\right)$$

$$= \frac{1}{4T}\int_0^T dt\left(e^{i(\delta_e+\delta_d-2\omega t)}+e^{i(2\omega t-\delta_e-\delta_d)}+e^{i(\delta_e-\delta_d)}+e^{i(\delta_d-\delta_e)}\right)$$

$$= \frac{1}{4T}\int_0^T dt\left[2\cos(\delta_e+\delta_d-2\omega t)+2\cos(\delta_e-\delta_d)\right]$$

$$\frac{1}{T}\int_0^T \cos(At+\phi) = 0$$

$$\langle E D \rangle = \frac{1}{2T}\cos(\delta_e-\delta_d)\int_0^T dt$$

$$= \frac{1}{2}\cos(\delta_e-\delta_d)$$

b) $\frac{1}{2}\mathrm{Re}(E D^*) = \frac{1}{2}\mathrm{Re}\, e^{i(kx-\omega t+\delta_e)}e^{-i(kx-\omega t+\delta_d)}$

$$= \frac{1}{2}\mathrm{Re}\, e^{i(\delta_e-\delta_d)}$$

$$= \frac{1}{2}\cos(\delta_e-\delta_d)$$

c) $\frac{1}{2}\mathrm{Re}\, E^* D = \frac{1}{2}\mathrm{Re}\, e^{-i(kx-\omega t+\delta_e)}e^{i(kx-\omega t+\delta_d)}$

$$= \frac{1}{2}\cos(\delta_d-\delta_e)$$

$$= \frac{1}{2}\cos(\delta_e-\delta_d)$$

d) $\langle u \rangle = \frac{1}{4}\mathrm{Re}\left(\epsilon_0 \vec{E}\cdot\vec{E}^* + \frac{1}{\mu_0}\vec{B}\cdot\vec{B}^*\right)$ $\quad \langle \vec{S} \rangle = \frac{1}{2\mu_0}(\vec{E}\times B^*)$