1.14 ⋆ Prove that for any two vectors **a** and **b**,

$$|\mathbf{a}+\mathbf{b}| \le (a+b).$$

[*Hint:* Work out $|\mathbf{a}+\mathbf{b}|^2$ and compare it with $(a+b)^2$.] Explain why this is called the triangle inequality.

1.15 ⋆ Show that the definition (1.9) of the cross product is equivalent to the elementary definition that $\mathbf{r}\times\mathbf{s}$ is perpendicular to both **r** and **s**, with magnitude $rs\sin\theta$ and direction given by the right-hand rule. [*Hint:* It is a fact (though quite hard to prove) that the definition (1.9) is independent of your choice of axes. Therefore you can choose axes so that **r** points along the x axis and **s** lies in the xy plane.]

1.16 ⋆⋆ **(a)** Defining the scalar product $\mathbf{r}\cdot\mathbf{s}$ by Equation (1.7), $\mathbf{r}\cdot\mathbf{s}=\sum r_i s_i$, show that Pythagoras's theorem implies that the magnitude of any vector **r** is $r=\sqrt{\mathbf{r}\cdot\mathbf{r}}$. **(b)** It is clear that the length of a vector does not depend on our choice of coordinate axes. Thus the result of part (a) guarantees that the scalar product $\mathbf{r}\cdot\mathbf{r}$, as defined by (1.7), is the same for any choice of orthogonal axes. Use this to prove that $\mathbf{r}\cdot\mathbf{s}$, as defined by (1.7), is the same for any choice of orthogonal axes. [*Hint:* Consider the length of the vector $\mathbf{r}+\mathbf{s}$.]

1.17 ⋆⋆ **(a)** Prove that the vector product $\mathbf{r}\times\mathbf{s}$ as defined by (1.9) is distributive; that is, that $\mathbf{r}\times(\mathbf{u}+\mathbf{v})=(\mathbf{r}\times\mathbf{u})+(\mathbf{r}\times\mathbf{v})$. **(b)** Prove the product rule

$$\frac{d}{dt}(\mathbf{r}\times\mathbf{s})=\mathbf{r}\times\frac{d\mathbf{s}}{dt}+\frac{d\mathbf{r}}{dt}\times\mathbf{s}.$$

Be careful with the order of the factors.

1.18 ⋆⋆ The three vectors **a**, **b**, **c** are the three sides of the triangle ABC with angles α, β, γ as shown in Figure 1.15. **(a)** Prove that the area of the triangle is given by any one of these three expressions:

$$\text{area}=\tfrac{1}{2}|\mathbf{a}\times\mathbf{b}|=\tfrac{1}{2}|\mathbf{b}\times\mathbf{c}|=\tfrac{1}{2}|\mathbf{c}\times\mathbf{a}|.$$

(b) Use the equality of these three expressions to prove the so-called law of sines, that

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}.$$

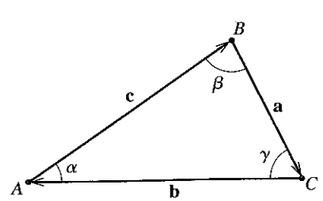

Figure 1.15 Triangle for Problem 1.18.

1.19 ⋆⋆ If **r**, **v**, **a** denote the position, velocity, and acceleration of a particle, prove that

$$\frac{d}{dt}[\mathbf{a}\cdot(\mathbf{v}\times\mathbf{r})]=\dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}).$$

1.20 ⋆⋆ The three vectors **A**, **B**, **C** point from the origin O to the three corners of a triangle. Use the result of Problem 1.18 to show that the area of the triangle is given by

$$(\text{area of triangle})=\tfrac{1}{2}|(\mathbf{B}\times\mathbf{C})+(\mathbf{C}\times\mathbf{A})+(\mathbf{A}\times\mathbf{B})|.$$

1.21 ⋆⋆ A parallelepiped (a six-faced solid with opposite faces parallel) has one corner at the origin O and the three edges that emanate from O defined by vectors **a**, **b**, **c**. Show that the volume of the parallelepiped is $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$.

1.22 ⋆⋆ The two vectors **a** and **b** lie in the xy plane and make angles α and β with the x axis. **(a)** By evaluating $\mathbf{a}\cdot\mathbf{b}$ in two ways [namely using (1.6) and (1.7)] prove the well-known trig identity

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.$$

(b) By similarly evaluating $\mathbf{a}\times\mathbf{b}$ prove that

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta.$$

1.23 ⋆⋆ The unknown vector **v** satisfies $\mathbf{b}\cdot\mathbf{v}=\lambda$ and $\mathbf{b}\times\mathbf{v}=\mathbf{c}$, where λ, **b**, and **c** are fixed and known. Find **v** in terms of λ, **b**, and **c**.

SECTION 1.4 Newton's First and Second Laws; Inertial Frames

1.24 ⋆ In case you haven't studied any differential equations before, I shall be introducing the necessary ideas as needed. Here is a simple exercise to get you started: Find the general solution of the first-order equation $df/dt=f$ for an unknown function $f(t)$. [There are several ways to do this. One is to rewrite the equation as $df/f=dt$ and then integrate both sides.] How many arbitrary constants does the general solution contain? [Your answer should illustrate the important general theorem that the solution to any nth-order differential equation (in a very large class of "reasonable" equations) contains n arbitrary constants.]

1.25 ⋆ Answer the same questions as in Problem 1.24, but for the differential equation $df/dt=-3f$.

1.26 ⋆⋆ The hallmark of an inertial reference frame is that any object which is subject to zero net force will travel in a straight line at constant speed. To illustrate this, consider the following: I am standing on a level floor at the origin of an inertial frame $\mathcal{S}$ and kick a frictionless puck due north across the floor. **(a)** Write down the x and y coordinates of the puck as functions of time as seen from my inertial frame. (Use x and y axes pointing east and north respectively.) Now consider two more observers, the first at rest in a frame $\mathcal{S}'$ that travels with constant velocity v due east relative to $\mathcal{S}$, the second at rest in a frame $\mathcal{S}''$ that travels with constant *acceleration* due east relative to $\mathcal{S}$. (All three frames coincide at the moment when I kick the puck, and $\mathcal{S}''$ is at rest relative to $\mathcal{S}$ at that same moment.) **(b)** Find the coordinates x', y' of the puck and describe the puck's path as seen from $\mathcal{S}'$. **(c)** Do the same for $\mathcal{S}''$. Which of the frames is inertial?

1.27 ⋆⋆ The hallmark of an inertial reference frame is that any object which is subject to zero net force will travel in a straight line at constant speed. To illustrate this, consider the following experiment: I am