Electromagnetic Theory I

Problem Set 4

Due: 30 September 2020

13. We have derived the spherical harmonic $Y_{ll}(\theta,\varphi)$,

$$Y_{ll}(\theta,\varphi) \;=\; \frac{(-)^l}{l!\sqrt{2\pi}}\sqrt{\frac{(2l+1)!}{2^{2l+1}}}\;\sin^l\theta e^{il\varphi}, \tag{1}$$

normalized so that $\int d\Omega|Y_{lm}|^2 = 1$, as a solution of the differential equation $L_+Y_{ll} = 0$ where $L_\pm = L_x \pm iL_y$ are the angular momentum ladder operators. Recall that $L_\pm$ raise or lower the m value of a spherical harmonic Y_{lm} by one unit:

$$L_\pm Y_{lm} \;=\; \sqrt{l(l+1)-m(m\pm 1)}Y_{lm\pm1} = \sqrt{(l\mp m)(l\pm m+1)}Y_{lm\pm1}\;. \tag{2}$$

Using these facts, apply L_- repeatedly to Y_{ll} to derive all the Y_{lm} for $0\le m\le l$ for the cases a) $l=1$, b) $l=2$, and c) $l=3$. You may use without proof the spherical coordinate form of the $L_\pm, L_z$:

$$L_\pm \;=\; \pm e^{\pm i\varphi}\left(\frac{\partial}{\partial\theta}\pm i\cot\theta\frac{\partial}{\partial\varphi}\right), \qquad L_z = \frac{1}{i}\frac{\partial}{\partial\varphi} \tag{3}$$

14. One can sometimes use part of the solution of a simple potential problem as the solution of an apparently more complicated problem. For example, any equipotential surface of the simple problem can be replaced by a conductor of coincident shape.

a) First, find the potential everywhere outside an isolated conducting sphere of radius R carrying charge Q, immersed in a uniform external electric field. (Hint: Take the z-axis parallel to the external field, expressing its potential in spherical coordinates, taking the center of the sphere at the origin.)

b) Now consider a hemispherical conductor of radius R attached to a grounded plane (the xy-plane) (i.e. the hemisphere-plane combination are all at zero potential). The top of the hemisphere is at $z=R$ and its center is at the origin of coordinates. The electric field, in the (empty) region above this conducting surface, approaches a uniform field $E_0\hat{z}$ far from the hemisphere. Calculate the surface charge density everywhere on the conducting surface (plane and hemisphere).

c) Calculate the total charge on the hemisphere of part b), in terms of E_0 and R.

15. J, Problem 3.4.

16. J, Problem 3.14.