Electromagnetic Theory I

Problem Set 6

Due: 14 October 2020

21. a) Use the power series definition of the Bessel function

$$J_m(x) \equiv \sum_{k=0}^{\infty} \frac{(-)^k}{k!\Gamma(k+1+m)} \left(\frac{x}{2}\right)^{2k+m} \tag{1}$$

to derive the recursion formulas for Bessel functions:

$$J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x), \qquad J_{m-1}(x) - J_{m+1}(x) = 2\frac{dJ_m(x)}{dx} \tag{2}$$

b) Explain why these same recursion formulas are valid for $N_m, H_m^{(1)}, H_m^{(2)}$.

c) Using the definitions

$$I_m(x) = i^{-m} J_m(ix), \qquad K_m(x) = \frac{\pi i^{m+1}}{2} H_m^{(1)}(ix) \tag{3}$$

obtain the analogous recursion formulas for I_m, K_m.

22. In class, we obtained the empty space Green function in cylindrical coordinates in the form

$$\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|} = \frac{1}{4\pi} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk J_{|m|}(k\rho) J_{|m|}(k\rho') e^{im(\varphi-\varphi')} e^{-k|z-z'|} \tag{4}$$

a) By adjusting the z dependent factors in this formula obtain the Dirichlet Green function for the region between the two infinite parallel planes $z = 0$ and $z = L$.

b) Use this Green function to calculate the potential between the planes when the plane $z = 0$ is held at zero potential and on the plane $z = L$ the potential is $\phi = V$ for $\rho \leq a$ and $\phi = 0$ for $\rho \geq a$. Your answer will be an integral over k.

c) Check that your answer reduces to the expected results when $a \to \infty$ at fixed a, ρ, z and also when $L \to \infty$ at fixed $a, \rho, L - z$.

23. Setting $z' = h < L$ and $\rho' = 0$, we know that q/ϵ_0 times the Green function obtained in part a) of the previous problem is just the potential for the system of two infinite grounded conducting planes with a charge q placed on the z axis at $z = h$.

a) Calculate the surface charge density induced on the upper plate as a function of ρ.

b) Using Green's reciprocation theorem with the situation of the previous problem 22 as a comparison problem, show that the induced charge within a radius $\rho \leq a$ on the upper plate is $Q(a) = -q\phi'(h, 0)/V$ where ϕ' is the potential you found in part b) of problem 22. From this result determine the total charge induced on the upper plate.

c) We also obtained an alternative expression for the empty space Green function obtained in cylindrical coordinates:

$$\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|} = \frac{1}{(2\pi)^2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk I_{|m|}(|k|\rho_<) K_{|m|}(|k|\rho_>) e^{im(\varphi-\varphi')} e^{ik(z-z')}. \quad (5)$$

By adjusting the z-dependent factor obtain the corresponding alternate Dirichlet Green function for the region between two parallel planes. Note that the adjustment includes replacing the integral over k by a discrete sum!

d) Using part c) write an alternate expression for the potential of this problem and calculate the surface charge density induced on both plates. Integrate to find the total induced charge and compare to that obtained in part b).

24. Solve J, Problem 3.22.