Electromagnetic Theory I

Problem Set 7

Due: 28 October 2020

25. J, Problem 4.1

26. We have seen that in Cartesian basis the 2^l pole moment is a rank l tensor $Q^{i_1 i_2 \cdots i_l}$, which is completely symmetric in its indices and traceless in every pair of indices. In this problem we repeat the counting we did in class in terms of the generating function defined in part a) below and then find the generalization to d dimensions.

a) Show that any rank l tensor with complete symmetry has $N_l = (l+1)(l+2)/2$ independent components by finding the generating function $p_3(x) = \sum_{l=0}^{\infty} N_l x^l$. Since Q is symmetric in all its indices, all components in which n_1 indices have the value 1, n_2 indices have the value 2 and n_3 indices have the value 3 are equal. Thus the number of independent components is the number of ways you can partition l into the sum of three integers $l = n_1 + n_2 + n_3$ where each n_k is a nonnegative integer $\leq l$. Using these facts, prove that $p_3(x) = (1-x)^{-3}$ and obtain the desired result by developing $p(x)$ in a Taylor series.

b) The tracelessness condition gives N_{l-2} relations among these components. Use this counting to prove that $Q^{i_1 i_2 \cdots i_l}$ has $2l+1$ independent components.

c) Now generalize to an arbitrary spatial dimension d in which you can show that $p_d(x) = (1-x)^{-d}$. Find the number of components of a symmetric tensor first, and then by subtracting find the number of components of a traceless one.

27. J, Problem 4.4

28. We have found that the method of images with a single image charge can give the potential for a point charge outside a grounded conducting sphere, and also the fields for a charge above the planar interface between two different homogeneous dielectrics. But as we shall see, the method falls short for a point charge outside a uniform dielectric sphere. Assume the dielectric constant is ϵ inside a sphere of radius R and ϵ_0 outside the sphere.

a) Set up and solve for the potential of a point charge q a distance $D > R$ from the center of the dielectric sphere as an expansion in Legendre polynomials (or spherical harmonics). You will have different expansion coefficients inside and outside the sphere, which you are to determine by the matching conditions at the boundary.

b) Show that in the limit $\epsilon/\epsilon_0 \to \infty$, the result reduces to the potential for a point charge outside a conducting sphere, by expanding the latter solution in Legendre polynomials and comparing coefficients.

c) For finite ϵ/ϵ_0 discuss why the solution cannot be expressed in terms of a single image charge, even though when $\epsilon/\epsilon_0 \to \infty$ it can.