Electromagnetic Theory II

Problem Set 2

Due: 27 January 2021

5. Starting with the formula for combining the velocities $v\hat{x}, \boldsymbol{V}$ relativistically

$$U_x = \frac{v + V_x}{1 + vV_x/c^2}, \qquad U_y = \frac{V_y}{\gamma(1 + vV_x/c^2)}, \qquad U_z = \frac{V_z}{\gamma(1 + vV_x/c^2)} \tag{1}$$

Show that $\boldsymbol{U}^2$ is always less than c^2, as long as the individual velocities $v\hat{x}, \boldsymbol{V}$ are less than c. Discuss the limits when one or both approach that of light.

6. We can form two scalar fields from the field strength tensor $F_{\mu\nu}(x)$;

$$\begin{aligned} I_1(x) &= \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(c^2\boldsymbol{B}^2 - \boldsymbol{E}^2), \qquad I_2(x) = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = -2c\boldsymbol{E}\cdot\boldsymbol{B} \\ I'_{1,2}(x') &= I_{1,2}(x) = I_{1,2}(\Lambda^{-1}x') \end{aligned}$$

If the fields are uniform in space-time, $I_{1,2}$ are just numbers which have the same values in all inertial frames. Their values control whether it is possible to find certain kinds of special frames.

a) What criteria on uniform static $\boldsymbol{E}, \boldsymbol{B}$ must be met if we are to find a frame in which one or the other or both are zero? Are there any fields which are purely electric in one inertial frame and purely magnetic in another inertial frame?

b) Suppose in the lab frame the angle between static uniform electric and magnetic fields $\boldsymbol{E}, \boldsymbol{B}$ with $c|\boldsymbol{B}| > |\boldsymbol{E}|$ is $\theta \neq 0, \pi/2$. (For definiteness take axes so $\boldsymbol{E}$ is parallel to the x-axis and $\boldsymbol{B}$ lies in the xy-plane.) Determine an inertial frame in which $\boldsymbol{E}$ and $\boldsymbol{B}$ are parallel.

c) What are the fields in the frame determined in b) in the case θ is very small, and in the case θ is very close to $\pi/2$?

7. Consider a point charge q moving along the positive x-axis at velocity v. In the primed frame moving with the particle, the electric field is just the static Coulomb field and the magnetic field is zero.

a) By applying the Lorentz transformation to these fields in the primed frame, show that in the unprimed frame the fields are

$$\boldsymbol{E} = \frac{q(\boldsymbol{r} - vt\hat{x})}{4\pi\epsilon_0\gamma^2[(x-vt)^2 + (y^2+z^2)(1-v^2/c^2)]^{3/2}}, \qquad \boldsymbol{B} = \frac{v}{c^2}\hat{x}\times\boldsymbol{E}$$

b) Notice that $\boldsymbol{E}$ is directed radially from the present location of the charge, but its strength is not isotropic. Compare the electric field at a large distance R from the charge along the x axis to the field at the same large distance R from the charge along the y (or z) axis. What happens to their ratio for v very close to the speed of light?

8. J, Problem 11.18. Please do this problem in SI units. Then the quoted electric field in part a) should have the right side multiplied by $1/4\pi\epsilon_0$, and the quoted magnetic field should be multiplied by $1/4\pi c\epsilon_0$. The potentials quoted in part c) will acquire similar factors.