Electromagnetic Theory II

Problem Set 3

Due: 3 February 2021

9. Consider the Lorentz transformation properties of the 4-vector current field $J^\mu = (c\rho, \boldsymbol{J})$. Let the primed frame move with velocity $\boldsymbol{v} = v\hat{x}$, parallel to the x-axis, relative to the unprimed frame.

a) Write out the Lorentz transformation which expresses $J^\mu(x)$ in terms of $J'(x') = J'(\Lambda x)$

b) Suppose the current density $\boldsymbol{J}' = 0$ and the charge density ρ' in the primed frame is static, has 0 total charge, but has a non zero electric dipole moment $\boldsymbol{p}'$. By working out the charge and current densities in the unprimed frame, find the electric ($\boldsymbol{p}$) and magnetic ($\boldsymbol{m}$) dipole moments in the unprimed frame. Show the limiting cases of $v << c$, where you should find $\boldsymbol{m} \approx \boldsymbol{p} \times \boldsymbol{v}/2$.

c) Now do the case where the charge density $\rho' = 0$ and the current density $\boldsymbol{J}'$ is static and such as to produce a magnetic moment $\boldsymbol{m}'$. Find the electric and magnetic moments and their nonrelativistic limits, where you should find $\boldsymbol{p} \approx \boldsymbol{v} \times \boldsymbol{m}/c^2$.

Be sure to use SI units, and to take account of the conditions on the moments of $\boldsymbol{J}$ following from $\nabla \cdot \boldsymbol{J} = 0$!

10. J, Problem 11.23

11. J, Problem 11.25

12. As we discussed in class, solving the motion of a particle in uniform electric and magnetic fields is straightforward, but tedious. The equations of motion

$$\frac{dU_\mu}{d\tau} = \frac{q}{mc} F_{\mu\nu} U^\nu \tag{1}$$

form a set of four coupled first order differential equations with constant coefficients for the four quantities $U_{0,1,2,3}$, which can be solved by first assuming all components have the same exponential time dependence $U_\mu = K_\mu e^{r\tau}$ so the differential equations reduce to a set of 4 linear algebraic equations for the K^r_μ associated with each r, and the consistency conditions $\det(F - rI) = 0$ that restrict the values of r to the four solutions of a quartic equation. The general solution can be constructed as a linear combination of these 4 special solutions.

a) Write out these equations in the coordinate system where $\boldsymbol{E} = E\hat{x}$ and $\boldsymbol{B} = B(\hat{x}\cos\theta + \hat{y}\sin\theta)$. Here θ is the angle between $\boldsymbol{B}$ and $\boldsymbol{E}$.

b) In view of the result of problem 6b) in Set 2, we can find an inertial frame where $\theta = 0$. Thus we actually lose no information by solving only this case. So for $\theta = 0$ find the general solution to the equations of part a) for $U_{0,1,2,3}(\tau)$ and then for $x_{0,1,2,3}$ by integrating $dx_\mu/d\tau = U_\mu(\tau)$. Remember that U^μ is subject to the constraint $U_\mu U^\mu = -c^2$, which restricts the overall scale of the coefficients in this linear combination. For simplicity, set the initial conditions: $U^1(0) = 0$ and $x^0(0) = 0$.

c) Specialize your solutions to zero magnetic field ($B = 0$), assuming $x^2 = x^3 = 0$, and recover the solutions for uniform electric field discussed in class.