Electromagnetic Theory II

Problem Set 4

Due: 10 February 2021

13. J, Problem 12.7 **in SI units**. Note that in SI units, the inequalities mentioned in parts b) and c) are $p > qBa$ and $p < qBa/2$ respectively. Also, in these units, the electromagnetic momentum density is just $\boldsymbol{g} = \boldsymbol{D} \times \boldsymbol{B} \to \epsilon_0 \boldsymbol{E} \times \boldsymbol{B}$ in empty space.

14. J, Problem 12.9

15. Recall the Lagrangian density for scalar electrodynamics

$$\mathcal{L} = -\frac{\epsilon_0}{4} F_{\mu\nu} F^{\mu\nu} - (\partial_\mu + iQA_\mu)\phi^* (\partial^\mu - iQA^\mu)\phi - U(\phi^*\phi), \qquad F_{\mu\nu} = c(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

a) Derive the canonical energy momentum tensor for this Lagrangian defined as

$$T^{\mu\nu} = -\sum_i \partial^\mu \psi_i \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi_i)} + \eta^{\mu\nu}\mathcal{L}$$

where the ψ_i are the 6 independent fields A_ν, ϕ, ϕ^*. Notice that, as we found for the free electodynamics, $T^{\mu\nu}$ is not symmetric in its indices.

b) As we did in the free case, find the modification that makes the result of part a) symmetric. Prove that the modification maintains the conservation condition $\partial_\nu T^{\mu\nu} = 0$.

16. We will be using the scalar electrodynamics to discuss superconductivity. In this problem we specialize the energy density to a situation with axially symmetric fields which we will later need to discuss magnetic flux vortices in superconductors. We use cylindrical coordinates ρ, φ, z and assume z-independent fields.

a) Show that the potential $\boldsymbol{A} = \hat{\varphi} A(\rho)$ produces a magnetic field $\boldsymbol{B} = B(\rho)\hat{z}$, and find $B(\rho)$ in terms of $A(\rho)$.

b) For the scalar field we assume a form $\phi = f(\rho)e^{im\varphi}$, with m an integer. Because the fields are z-independent, the total energy is infinite, so it is appropriate to consider instead the energy per unit length $\int \rho d\rho d\varphi T^{00}$. Using the symmetric energy momentum tensor of the previous problem write out the energy per unit length of the magnetic vortex in terms of $A(\rho)$ and $f(\rho)$, expressed as a one dimensional integral over ρ.