Electromagnetic Theory II

Problem Set 5

Due: 17 February 2021

17. In problem 16 of set 4 you showed that the energy per unit length of a static and axially symmetric configuration $\boldsymbol{A} = \hat{\varphi}A(\rho)$, $\phi = f(\rho)e^{im\varphi}$ with m an integer, of electromagnetic and scalar fields was

$$T = 2\pi \int_0^\infty d\rho \left[\epsilon_0 c^2 \frac{[(\rho A)']^2}{2\rho} + \rho f'^2 + \rho \left(\frac{m}{\rho} - QA \right)^2 f^2 + \rho U(f^2) \right] \tag{1}$$

The total energy of course would be $\int dz T = \infty$. To describe a superconducting state U must have a minimum for $f \neq 0$. For this problem assume that $U(f^2) = \lambda(f^2 - f_0^2)^2/4$. Static solutions of the equations of motion with axial symmetry are stationary points of T. The lowest energy solution minimizes T. Thus we can use a variational principle to approximate the lowest energy solution for each fixed m. When $m \neq 0$ such a solution will describe a magnetic vortex.

a) Show that in order for T to be finite, the fields must have the behavior $f \to f_0$ and $\rho A \to m/Q$ as $\rho \to \infty$. Evaluate the total magnetic flux $\Phi_B = 2\pi \int_0^\infty \rho d\rho B(\rho)$ for such a vector potential. Notice that it is quantized, and give the fundamental unit.

b) At first glance it would seem that the choices $A = m/Q\rho$ and $f = f_0$ lead to a value of $T = 0$. However, this is too glib. Show that a vector potential of this form implies a magnetic field $\boldsymbol{B} = \Phi_B \delta(x)\delta(y)$. Such a field would contribute $\epsilon_0 c^2 \Phi_B^2 \delta(0)\delta(0)/2 = \infty$ to the energy per unit length. Confirm this conclusion by studying the first term in T for a regularized potential $A = m\rho/Q(\rho^2 + \delta^2)$, showing how it blows up as $\delta \to 0$. Accordingly a finite energy solution must satisfy $A \to 0$ as $\rho \to 0$, and then the third term will only be finite if also $f \to 0$ as $\rho \to 0$.

18. With simple trial functions for A, f that have the necessary behaviors at $\rho \to 0, \infty$, as determined in the previous problem, make a variational estimate of the energy per unit length T of a magnetic vortex. Discuss its dependence on λ, Q, f_0. (Some possible trials are $A = m\rho/(\rho^2 + \alpha^2)Q$ and $f^2 = f_0^2\rho^2/(\rho^2 + \beta^2)$, but feel free to invent your own trials.)

19. J, Problem 12.15. The Proca equation is just the Maxwell equation for the scalar electrodynamics we discussed in class, for the case of a constant scalar field $\phi = \phi_0$. $\epsilon_0 c \partial_\nu F^{\mu\nu} = -\mu^2 A^\mu$. The photon "mass" parameter is $\mu \equiv \sqrt{2Q^2|\phi_0|^2}$ in this problem. Please solve this problem using SI units throughout. This means that the quoted fields in a) and b) will contain appropriate factors of ϵ_0, μ_0, c, and 4π!

20. J, Problem 12.18.