Electromagnetic Theory II

Problem Set 8

Due: 17 March 2021

29. J, Problem 8.4. Circular cylindrical wave guide with finite σ

30. **Spherical cavity** In this problem we investigate some of the normal modes for electromagnetic fields with harmonic time dependence (*i.e.* $\propto e^{-i\omega t}$) in a *spherical* cavity of radius a in a perfect conductor. Use polar coordinates with origin at the center of the cavity. When $k \equiv \omega/c \neq 0$, the source-free Maxwell equations can be reduced in two equivalent ways to:

$$\begin{aligned} \text{I}: \qquad & \boldsymbol{B} = \frac{1}{ikc}\nabla\times\boldsymbol{E} \qquad \nabla\cdot\boldsymbol{E}=0 \qquad (-\nabla^2-k^2)\boldsymbol{E}=0 \\ \text{or}\quad \text{II}: \qquad & \boldsymbol{E} = -\frac{c}{ik}\nabla\times\boldsymbol{B} \qquad \nabla\cdot\boldsymbol{B}=0 \qquad (-\nabla^2-k^2)\boldsymbol{B}=0 \end{aligned}$$

a) For $\omega \neq 0$ the fields must be strictly zero in the bulk of the conductor. Why? Use Maxwell's equations to prove that the fields in the cavity must then satisfy the boundary conditions $\boldsymbol{E}_t = \boldsymbol{B}_n = 0 \quad \text{at } r = a$.

b) Consider the following forms for TE ($\hat{\mathbf{r}}\cdot\boldsymbol{E} = 0$) and TM ($\hat{\mathbf{r}}\cdot\boldsymbol{B} = 0$) modes:

$$\text{TE}: \quad \boldsymbol{E}_{\text{TE}} = f(r)\hat{\mathbf{r}}\times\boldsymbol{C}e^{-i\omega t}; \qquad \text{TM}: \quad \boldsymbol{B}_{\text{TM}} = g(r)\hat{\mathbf{r}}\times\boldsymbol{D}e^{-i\omega t},$$

where $\boldsymbol{C}$ and $\boldsymbol{D}$ are constant vectors. It is clearly convenient to use form (I), (II) of Maxwell's equations for the TE, TM cases respectively. Show that $\boldsymbol{E}_{TE}$, $\boldsymbol{B}_{TM}$ automatically have zero divergence and that all boundary conditions will be satisfied in the TE case if $f(a) = 0$ and in the TM case if $(rg(r))'|_{r=a} = 0$.

c) In the TE case show that last equation in (I) will be satisfied provided $f(r)$ satisfies the differential equation

$$\left(-\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{2}{r^2} - k^2\right) f = 0,$$

where $k = \omega/c$. Obviously the TM case works the same way using form (II) if g satisfies the same equation. *Hint*: Recall that the components of $\hat{\mathbf{r}}$ can be expressed as linear combinations of the spherical harmonics $Y_{1,\pm1}(\theta,\phi), Y_{1,0}(\theta,\phi)$, and $-\nabla^2 Y_{lm}(\theta,\phi) = (l(l+1)/r^2)Y_{lm}(\theta,\phi)$.

d) Show that the spherical Bessel function

$$j_1(kr) \equiv \frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}$$

is the unique solution of this differential equation that is regular at $r = 0$. Imposing the appropriate boundary condition, obtain a graphical solution for the lowest frequency in each case. Which frequency is lower, TE or TM?

31. J, Problem 8.6 Circular cylindrical cavity.

32. J, Problem 8.9 Variational method for cavity modes.