Electromagnetic Theory II

Problem Set 11

Due: 7 April 2021

40. Consider a linear antenna of length d for which the current density is given to be $\boldsymbol{J} = I\hat{z}\delta(x)\delta(y)\sin(2\pi z/d)e^{-i\omega t}$ for $-d/2 < z < d/2$, so the current density vanishes at the endpoints and at the midpoint of the antenna.

a) Evaluate the Fourier transform $\int d^3x \boldsymbol{J} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}$ exactly and use it to determine the exact power angular distribution $dP/d\Omega$ for any k. Check that for wavelength $2\pi/k = d$ your answer simplifies to

$$\frac{dP}{d\Omega}_{k=2\pi/d} = \frac{Z_0 I^2}{8\pi^2}\frac{\sin^2[\pi\cos\theta]}{\sin^2\theta} \tag{1}$$

b) Now apply our vector spherical harmonics method to this current density. Show that all the magnetic multipole coefficients $a^M_{lm} = 0$, and by expressing the electric muktipole coefficients a^E_{lm} as an integral over z with $0 < z < d/2$ show that only the ones with l even and $m = 0$ are nnvanishing.

c) For the special value of $k = 2\pi/d$, notice that the integrand of part b) is a total derivative, and so evaluate all of the nonvanishing a^E exactly.

d) For $k = 2\pi/d$, evaluate the contribution of the lowest nonvanishing a^E to $dP/d\Omega$ and compare to the exact result of part a). Plot both distributions on the same graph.

41. The electric field of a plane wave $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ is perpendicular to the propagation direction $\boldsymbol{k}$. The polarization vector $\boldsymbol{\varepsilon}$ is by definition a unit vector parallel to the electric field. We have found it useful to describe the fields with complex vectors whose real parts are the physical fields. In this description polarization vectors can be complex when they describe elliptic polarization, in which case we adopt the normalization $\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon} = 1$. Complex vectors belong to a three dimensional complex vector space, and it is convenient to pick an orthonormal basis. Let $\boldsymbol{n} = \hat{k}$ be the unit vector parallel to $\boldsymbol{k}$. Then choose $\boldsymbol{e}_3 = \boldsymbol{n}$, $\boldsymbol{e}_1 = \boldsymbol{\varepsilon}$, and $\boldsymbol{e}_2 = \boldsymbol{n}\times\boldsymbol{\varepsilon}^*$.

a) Show that the three $\boldsymbol{e}_a$ are orthonormal in the sense that $\boldsymbol{e}^*_a\cdot\boldsymbol{e}_b = \delta_{ab}$, and prove the completeness relation $\sum_{a=1}^3 e^i_a e^{j*}_a = \delta_{ij}$. Note that the completeness relation can be rearranged as

$$\sum_{a=1}^{2} e^i_a e^{j*}_a = \delta_{ij} - e^i_3 e^{j*}_3 = \delta_{ij} - n^i n^j$$

which is useful to sum cross sections over polarizations, $\sum_{\text{pol}} \epsilon^i\epsilon^{j*} = \delta_{ij} - n^i n^j$.

b) We can expand any real unit vector in three space $\boldsymbol{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$ in the new basis $\boldsymbol{v} = \sum_a V_a\boldsymbol{e}_a$. Show that $\sum_a |V_a|^2 = 1$

c) Prove the identity

$$|\hat{r}\cdot\boldsymbol{n}|^2 + |\hat{r}\cdot\boldsymbol{\epsilon}|^2 + |\hat{r}\cdot(\boldsymbol{n}\times\boldsymbol{\varepsilon})|^2 = 1$$

where $\hat{r}$ is the radial unit vector.

42. J, Problem 10.1. Hint: apply the results of problem 41 to the result in Eq. (10.14).
43. J, Problem 10.2. Again it is sufficient to work with Eq. (10.14).