8 Lorentz Invariance and Special Relativity

The principle of special relativity is the assertion that all laws of physics take the same form as described by two observers moving with respect to each other at constant velocity $\boldsymbol{v}$. If the dynamical equations for a system preserve their form under such a change of coordinates, then they must show a corresponding symmetry. Newtonian mechanics satisfied this principle in the form of Galilei relativity, for which the relation between the coordinates was simply $\boldsymbol{r}'(t) = \boldsymbol{r}(t) - \boldsymbol{v}t$, and for which time in the two frames was identical. However, Maxwell's field equations do not preserve their form under this change of coordinates, but rather under a modified transformation: the Lorentz transformations.

8.1 Space-time symmetries of the wave equation

Let us first study the space-time symmetries of the wave equation for a field component in the absence of sources:

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi \;=\; 0 \tag{411}$$

As we discussed last semester spatial rotations $x'^k = R_{kl}x^l$ are realized by the field transformation $\psi'(\boldsymbol{x}', t) = \psi(\boldsymbol{x}, t) = \psi(R^{-1}\boldsymbol{x}', t)$. Then $\nabla'_k\psi' = R^{-1}_{lk}\nabla_l\psi = R_{kl}\nabla_l\psi$, and the wave equation is invariant under rotations because ∇^2 is a rotational scalar. If we have set up a fixed Cartesian coordinate system we may build up any rotation by a sequence of rotations about any of the three axes. Instead of specifying the axis of one of these basic rotations, it is more convenient to specify the plane in which the coordinate axes rotate. For example, we describe a rotation by angle θ about the z-axis as a rotation in the xy-plane.

$$\nabla'_x = \cos\theta\nabla_x - \sin\theta\nabla_y, \qquad \nabla'_y = \sin\theta\nabla_x + \cos\theta\nabla_y. \tag{412}$$

Then it is easy to see from the properties of trig functions that

$$\nabla'^2_x + \nabla'^2_y + \nabla'^2_z = \nabla^2_x + \nabla^2_y + \nabla^2_z, \tag{413}$$

under a rotation in any of the three planes, and through composition under any spatial rotation.

There must be a similar symmetry in the xt-, yt-, and zt-planes. But because of the relative minus sign we have to use hyperbolic trig functions instead of trig functions:

$$\nabla'_x = \cosh\lambda\nabla_x + \sinh\lambda\frac{\partial}{c\partial t}, \qquad \frac{\partial}{c\partial t'} = +\sinh\lambda\nabla_x + \cosh\lambda\frac{\partial}{c\partial t} \tag{414}$$

The invariance of $\nabla^2_x - \partial^2/c^2\partial t^2$ under this transformation then follows. Clearly there are analogous symmetries in the yt- and zt-planes. These transformations will replace the Galilei boosts of Newtonian relativity.

Now let us interpret this symmetry in terms of how the coordinates transform:

$$x' = x\cosh\lambda - ct\sinh\lambda \equiv \gamma(x - vt) \tag{415}$$

$$ct' = ct\cosh\lambda - x\sinh\lambda \equiv \gamma(ct - vx/c) \tag{416}$$

where $v = c\tanh\lambda$. From the identity $\cosh^{-2} = 1 - \tanh^2$, we see that $\gamma \equiv \cosh\lambda = 1/\sqrt{1 - v^2/c^2}$. We see from the first equation that the origin of the primed coordinate system $x' = 0$ corresponds to $x = vt$ which means that the origin of the primed system is moving on the x-axis at the speed v. We say that the primed system is boosted by velocity $\boldsymbol{v} = v\hat{x}$ with respect to the unprimed system. It is also essential for the invariance of the wave equation that the time t transform as shown.

The three rotations in the xy-, yz-, and zx-plane together with the three boosts in the xt-, yt-, and zt-planes generate the group of Lorentz transformations. If the only entities in the physical world were fields satisfying the wave equation, we would have to identify the Lorentz boosts with transformations to new inertial frames. As we shall see, Maxwell's equations are also invariant under Lorentz transformations, provided that the electric and magnetic fields are appropriately transformed among themselves. Indeed we already know that the electric and magnetic fields transform as vectors under ordinary rotations. To discuss their transformation under boosts we will have to develop the formalism of 4-vectors and 4-tensors.

8.2 Einstein's Insights

Einstein's contribution to special relativity was not the discovery of Lorentz transformations (which were already well-known). Rather, he recognized that the clash between Lorentz invariance and the Galilei invariance of Newtonian mechanics was inconsistent with the physical principle of relativity. The laws of Maxwell electrodynamics and Newtonian mechanics cannot both look the same to different inertial observers, since they require different transformation rules. One or the other required modification, and Einstein realized that the defect was in Newtonian mechanics, for which inertial frames are related by Galilei transformations

$$\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{v}t, \qquad t' = t \tag{417}$$

For $v << c$ Lorentz transformations are indistinguishable from Galilei transformations, which accounts for the lack of experimental evidence at the time that Newtonian mechanics was wrong. On the other hand experiments explicitly designed to show that Maxwell's equations took different forms in different inertial frames had repeatedly obtained null results. Einstein recognized this as direct evidence that Lorentz transformations gave the correct relationship between inertial frames. If this is the case, Newtonian mechanics has to be modified. Accordingly, Einstein developed a new relativistic (i.e. Lorentz invariant) particle mechanics.

An alternative way to discover the necessary modifications to Newtonian mechanics is to describe massive matter with Lorentz invariant wave equations. Unlike light, which always travels at speed c, material bodies can be brought to rest. The wave equation for light describes wave packets with definite wave number with group velocity $\boldsymbol{v}_g = c\hat{k}$, which always has magnitude c. However consider the (Klein-Gordon) wave equation

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \mu^2\right)\psi = 0 \tag{418}$$

Then a plane wave $e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t}$ will solve this equation if $\omega^2 = c^2(\boldsymbol{k}^2 + \mu^2)$, so the group velocity is $\boldsymbol{v}_g = c\boldsymbol{k}/\sqrt{\boldsymbol{k}^2 + \mu^2}$, which vanishes as $\boldsymbol{k} \to 0$. On the other hand the new wave equation is transparently Lorentz invariant, in the same way that the massless wave equation is.

Of course, in Einstein's time the idea that material bodies were wave packets must have seemed a little far-fetched. so he, instead, worked to directly modify the laws of particle mechanics. Ironically, Einstein was actually the first to realize that because of quantum mechanics electromagnetic waves display particle-like properties: the photon can be thought of as a plane wave with energy $E = \hbar\omega$ and momentum $\boldsymbol{p} = \hbar\boldsymbol{k}$. Applying this insight to our massive wave packet leads, with $\hbar\mu \equiv mc$, to

$$E = c\sqrt{\boldsymbol{p}^2 + m^2c^2} \sim mc^2 + \frac{\boldsymbol{p}^2}{2m} + O\left(\frac{\boldsymbol{p}^4}{m^3c^2}\right), \qquad \boldsymbol{v}_g = c\frac{\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2 + m^2c^2}} \tag{419}$$

where the last form on the right displays the low momentum limit $p \ll mc$. This shows that the Newtonian kinetic energy is obtained at low momentum and that m is the Newtonian mass. At $\boldsymbol{p} = 0$ we have Einstein's famous $E = mc^2$!

We can easily solve for $\boldsymbol{p}$ in terms of $\boldsymbol{v}$:

$$\frac{\boldsymbol{v}^2}{c^2} = \frac{\boldsymbol{p}^2}{\boldsymbol{p}^2 + m^2c^2}, \qquad E = \frac{mc^2}{\sqrt{1 - \boldsymbol{v}^2/c^2}} = \gamma mc^2, \qquad \boldsymbol{p} = \frac{\boldsymbol{v}E}{c^2} = \gamma m\boldsymbol{v} \tag{420}$$

As we shall soon see this modification of the relation between momentum and velocity is sufficient to make particle mechanics consistent with Lorentz invariance. In particular we shall see that with the Lorentz force law and this relation, Newton's equation of motion in the form

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{421}$$

is valid in all inertial frames connected by Lorentz transformations. But to see this clearly, we need to develop the machinery of 4-vectors and 4-tensors and their transformation laws.

8.3 Some Kinematical Aspects of Lorentz transformations

Time Dilatation Let us consider a clock moving down the x-axis according to $x(t) = vt, y(t) = z(t) = 0$. The clock is at rest at the origin of coordinates in the primed frame with spacetime coordinates given by

$$x' = \gamma(x - vt), \qquad t' = \gamma(t - vx/c^2), \qquad y' = y, \qquad z' = z \tag{422}$$

The time registered on the clock is t', which is related to the time in the unprimed frame by

$$t' = \gamma(v)(t - vx(t)/c^2) = \gamma(v)(t - v^2t/c^2) = t\sqrt{1 - v^2/c^2} \tag{423}$$

where we took into account the fact that the x coordinate of the clock is changing in the unprimed frame, $x(t) = vt$. Thus the observer in the unprimed frame judges the clock to run more slowly than in its rest frame by a factor of $1/\sqrt{1 - v^2/c^2} = \gamma(v)$.

Now consider a clock at rest at the origin in the unprimed frame. Now the relation between the times in the two frames is simply $t' = \gamma t$. And the primed observer also judges this clock to run more slowly by the same factor $\gamma(v)$. Each observer concludes that the other observer's clocks slow down! This effect is literally confirmed daily in accelerator experiments where fast unstable particles are able to travel distances many times their lifetime times c.

More generally, consider a clock in arbitrary motion, described by a trajectory $\boldsymbol{r}(t)$ in the unprimed frame. At a given moment we can consider a primed frame in which the clock is instantaneously at rest. Then assuming at that moment $\dot{\boldsymbol{r}} = v\hat{x}$ the Lorentz transformation of an infinitesimal time interval dt' to the Lab frame reads $dt' = \gamma(dt - vdx/c^2) = \gamma(1 - v^2/c^2)dt = dt\sqrt{1 - v^2/c^2}$. By rotational invariance, if $\dot{\boldsymbol{r}} = \boldsymbol{v}$ is in a general direction this implies $dt' = dt\sqrt{1 - \boldsymbol{v}^2/c^2}$. Then a finite lapse of time in the instantaneous rest frame $\Delta T = t_2' - t_1'$ is given by the integral

$$\Delta T = \int_{t_1}^{t_2} dt\sqrt{1 - \left(\frac{1}{c}\frac{d\boldsymbol{r}}{dt}\right)^2} < t_2 - t_1 \,. \tag{424}$$

The inequality confirms that a moving clock slows down no matter what the details of the motion. In particular the clock could start out at rest at the origin, accelerate to some speed, decelerate to rest, turn around accelerate and decelerate to arrive at rest where it started. The clock will show an elapsed time ΔT whereas a clock that stays at rest at the origin will show an elapsed time $t_2 - t_1 > \Delta T$. Because the clock is undergoing acceleration an observer moving with the clock is no longer an inertial observer (he is feeling all kinds of (fictitious) centrifugal forces), so there is no longer a symmetry between the moving observer and the one at rest.

Invariant intervals and the Light Cone Points in spacetime are more precisely thought of as events. By construction Lorentz transformations leave the quantity $x \cdot x = \boldsymbol{x}^2 - c^2t^2$ invariant. But since all events are subject to the same transformation, the "interval" between two events $s_{12}^2 = (x_1 - x_2) \cdot (x_1 - x_2)$ is also invariant. Intervals can be positive (space-like), negative (time-like) or zero (light-like). If one of the two events is at the origin $x_1^\mu = 0$, the events light-like separated from the origin lie on a double cone with vertices at the origin. Events within the cone have time-like separation from the origin, and those outside have space-like separation. There can be no causal connection between space-like separated events because they cannot be connected by light signals. We may define the proper time τ of a body in motion through $d\tau^2 = -dx \cdot dx/c^2 = dt^2 - d\boldsymbol{r}^2/c^2 = dt^2(1 - \dot{\boldsymbol{r}}^2/c^2)$. Note that a clock moving with the body registers the body's proper time.

Length Contraction Consider a rod of length L at rest in the unprimed frame parallel to the x-axis, with ends, say at $x_1 = 0$ and $x_2 = L$. In the primed system it is moving in the negative x direction at speed v: $x_1' = \gamma(-vt)$, $x_2' = \gamma(L - vt)$. Its length will be judged to be $L' = x_2'(t') - x_1'(t')$ where the coordinates are measured at the same t'. But these will correspond to different values of t: $t' = \gamma(t_1) = \gamma(t_2 - vL/c^2)$, so $t_2 = t_1 + Lv/c^2$. Then

$$L' = \gamma(L - vt_2) + \gamma vt_1 = \gamma L(1 - v^2/c^2) = L\sqrt{1 - v^2/c^2} \tag{425}$$

A moving object is therefor observed to be contracted by a factor $1/\gamma$ in the direction of motion.

We can also consider the rod at rest in the primed frame, in which case $x_1' = 0, x_2' = L$. At a particular time t in the unprimed frame these coordinates are related to the unprimed ones by $x_1' = \gamma(v)(x_1 - vt), x_2' = \gamma(v)(x_2 - vt)$, so $L = x_2' - x_1' = \gamma(v)(x_2 - x_1)$, so we reach the same conclusion, that length is perceived to be contracted by motion $x_2 - x_1 = L\sqrt{1 - v^2/c^2}$.

Simultaneity is relative Simultaneous events (x_1, t), (x_2, t) in the unprimed frame correspond to events (x_1', t_1'), (x_2', t_2') where $t_2' - t_1' = \gamma v(x_1 - x_2)/c^2$ in the primed frame. Of course two simultaneous events at the same spatial point are simultaneous in all frames.

Addition of velocities Suppose a particle is moving with constant velocity (V_x, V_y, V_z) in the primed frame, so $x' = V_x t'$, $y' = V_y t'$, $z' = V_z t'$. Then $x - vt = V_x(t - vx/c^2)$, $y = V_y\gamma(t - vx/c^2)$, $z = V_z\gamma(t - vx/c^2)$.

$$x = \frac{v + V_x}{1 + vV_x/c^2}t, \qquad y = \frac{V_y}{\gamma(1 + vV_x/c^2)}t, \qquad z = \frac{V_z}{\gamma(1 + vV_x/c^2)}t \tag{426}$$

Calling the components of velocity in the unprimed frame (U_x, U_y, U_z) we see that

$$U_x = \frac{v + V_x}{1 + vV_x/c^2}, \qquad U_y = \frac{V_y}{\gamma(1 + vV_x/c^2)}, \qquad U_z = \frac{V_z}{\gamma(1 + vV_x/c^2)} \tag{427}$$

$$\begin{aligned} \boldsymbol{U}^2 &= \frac{(v + V_x)^2 + (\boldsymbol{V}^2 - V_x^2)(1 - v^2/c^2)}{(1 + vV_x/c^2)^2} \\ &= c^2 - c^2\frac{(1 - \boldsymbol{V}^2/c^2)(1 - v^2/c^2)}{(1 + vV_x/c^2)^2} \end{aligned} \tag{428}$$

From the last form we see that $\boldsymbol{U}^2 \to c^2$ when either $v^2 \to c^2$ or $\boldsymbol{V}^2 \to c^2$ or both. Thus one never achieves a velocity greater than c by adding two velocities, no matter how close they are to c.

8.4 Space-time Tensors and their Transformation Laws

Let us introduce a unified notation x^μ for space-time coordinates where the index assumes the values $\mu = 0, 1, 2, 3$ and we identify $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$. The factor of c in x^0 assures that all components of x^μ have dimensions of length. Then a general Lorentz transformation can be expressed as a linear transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, where the 4×4 matrix $\Lambda^\mu{}_\nu$ is restricted by the requirement that $x \cdot x \equiv \boldsymbol{x}^2 - c^2t^2$ is invariant under the transformation. To express the consequences of this invariance, it is convenient to introduce the diagonal matrix $\eta_{\mu\nu}$, which has nonvanishing components $\eta_{11} = \eta_{22} = \eta_{33} = -\eta_{00} = 1$. Then we can write $x \cdot x = \eta_{\mu\nu}x^\mu x^\nu$, and the invariance condition becomes

$$x' \cdot x' = \eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho x^\sigma = \eta_{\rho\sigma}x^\rho x^\sigma = x \cdot x, \qquad \eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \tag{429}$$

An immediate consequence of the second equation is $(\det\Lambda)^2 = 1$ implying $\det\Lambda = \pm 1$. Notice that with the stated condition on Λ, it follows that $v' \cdot w' = v \cdot w$ for any v^μ, w^μ which transform like coordinates: $v'^\mu = \Lambda^\mu{}_\rho v^\rho$ and $w'^\mu = \Lambda^\mu{}_\rho w^\rho$.

Matrix Notation We can arrange the components of x^μ in a column vector

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^2 \end{pmatrix} \tag{430}$$

and the components of $\Lambda^\mu{}_\nu$ in a 4×4 matrix

$$\Lambda = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix}. \tag{431}$$

Then the equation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ becomes simply $x' = \Lambda x$, and the equation $\eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$ reads

$$\Lambda^T \eta \Lambda = \eta, \qquad \eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{432}$$

As an example a boost v in the x-direction and a rotation in the xy plane are given by

$$B_x = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad R_{xy} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{433}$$

respectively.

Contravariant Tensors We define a contravariant 4-vector v^μ as a 4 component entity whose components transform just like the coordinates $v'^\mu = \Lambda^\mu{}_\nu v^\nu$. We also stipulate that 4-vectors are unchanged under space-time translations. For this reason the four coordinates

x^μ are not components of a 4-vector, because they change under translations $x \to x + a$. But displacements, which are differences of coordinates are 4-vectors. In particular, the differentials dx^μ are contravariant 4-vectors.

Another important contravariant 4-vector is the energy momentum 4-vector of a particle $p^\mu = (E/c, \boldsymbol{p}) = \gamma m(c, \boldsymbol{v})$. The easiest way to see that it transforms as a 4-vector is to consider the boost which takes the particle from rest to velocity $\boldsymbol{v} = v\hat{x}$. Let the unprimed frame be the one where the particle is at rest $p^{1,2,3} = 0$ and $p^0 = mc$. Then the primed frame should move in the negative x direction with speed v. If p^μ is a contravariant 4-vector, its components in the new frame should then be

$$p'^0 = \gamma(p^0 + \frac{v}{c}p^1) = \gamma mc, \qquad p'^1 = \gamma(p^1 + \frac{v}{c}p^0) = \gamma mv \tag{434}$$

which are precisely the values we have found for a particle with this velocity. An ordinary rotation can then point the velocity in any direction. Incidentally, in our discussion of the Doppler shift we shall see that the frequency and wave number of a plane wave $k^\mu = (\omega/c, \boldsymbol{k})$ also transform like a 4-vector. This ensures that the Einstein/deBroglie quantum relations $p^\mu = \hbar k^\mu$ are compatible with Lorentz invariance.

Doppler Shift We examine how a plane wave $e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t}$ in the unprimed frame looks in the primed frame. Thinking of $k^\mu = (\omega/c, \boldsymbol{k})$ as a 4-vector, the plane wave can be written $e^{ik\cdot x} = e^{ik'\cdot x'}$. thus in the primed frame, we also have a plane wave but with frequency and wave numbers

$$\omega' = \gamma(\omega - vk^x), \qquad k'^x = \gamma(k^x - v\omega/c^2), \qquad k'^y = k^y \qquad k'^z = k^z \tag{435}$$

Writing $k^x = \omega\cos\theta/c$, $k^y = \omega\sin\theta\cos\varphi/c$, $k^z = \omega\sin\theta\sin\varphi/c$, and similarly for $k', \theta', \varphi' = \varphi$, these transformation laws lead to

$$\omega' = \gamma\omega(1 - \frac{v}{c}\cos\theta), \qquad \tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - v/c)} \tag{436}$$

If $\theta = 0$ the primed observer is moving away from the source of light and we see in this case $\omega' = \omega\sqrt{(c-v)/(c+v)} < \omega$ which means a red shift. On the other hand, if $\theta = \pi$ the observer is moving toward the source of light and we find a blue shift $\omega' = \omega\sqrt{(c+v)/(c-v)} > \omega$. Note that there is a relativistic Doppler effect when $\theta = \pi/2$ when $\omega' = \gamma\omega > \omega$. This transverse Doppler shift is a blue shift. In contrast, at $\theta' = \pi/2$, $\cos\theta = v/c$, and $\omega' = \omega\sqrt{1 - v^2/c^2}$, a red shift!

Covariant Tensors In special relativity, there is another type of 4-vector called a covariant 4-vector, which transforms as the 4-gradient operator $\partial_\mu \equiv \partial/\partial x^\mu$. To see that the transformation is different we use the chain rule to write

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = \Lambda^\nu{}_\mu \partial'_\nu, \qquad \partial'_\nu = \partial_\mu(\Lambda^{-1})^\mu{}_\nu \tag{437}$$

More generally any covariant vector v_μ transforms as $v'_\mu = v_\nu(\Lambda^{-1})^\nu{}_\mu$. We see two differences with the contravariant transformation law: first Λ^{-1} occurs and secondly the first rather than the second index is summed, i.e. the transpose occurs. Note that because of this difference the scalar $v_\mu w^\mu$ is invariant under all linear transformations, not just Lorentz transformations. The condition for ordinary rotations $R^{-1} = R^T$ obliterates the distinction between the two kinds of vectors. But for Lorentz transformations, we have instead the condition

$$\begin{aligned} \Lambda^{-1} &= \eta^{-1}\Lambda^T\eta \\ (\Lambda^{-1})^\mu{}_\nu &= \eta^{\mu\sigma}\Lambda^\rho{}_\sigma\eta_{\rho\nu} = \eta_{\nu\rho}\Lambda^\rho{}_\sigma\eta^{\sigma\mu} \end{aligned} \tag{438}$$

where $\eta^{\sigma\mu}$ has identical components to $\eta_{\sigma\mu}$ but we raise the indices because we are visualizing it as η^{-1}. Because of the -1 entries in η, covariant and contravariant 4-vectors have slightly different transformation laws. We also see from this exercise that a contravariant vector can be converted to a covariant one using the matrix η: $v_\mu \equiv \eta_{\mu\nu}v^\nu$. Or the other way around $v^\mu = \eta^{\mu\nu}v_\nu$. We see that $v^{1,2,3} = v_{1,2,3}$ but $v^0 = -v_0$. Although the distinction is very slight we shall maintain it throughout. The following forms for the scalar product of two four vectors summarize the distinctions we are making:

$$v \cdot w = \eta_{\mu\nu}v^\mu w^\nu = v^\mu w_\mu = v_\nu w^\nu = \eta^{\mu\nu}v_\mu w_\nu \tag{439}$$

From now on the summation convention for repeated indices will always involve the contraction of an upper (contravariant) index with a lower (covariant) index.

Once we have understood contravariant and covariant 4-vectors, multi-index tensors (e.g. $T^{\mu\nu}{}_\lambda$ are an easy generalization. Each contravariant (upper) index transforms as a contravariant vector and each covariant (lower) index like a covariant vector. A tensor field also has its space-time argument transformed:

$$T'^{\mu\nu}{}_\lambda(x') = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma(\Lambda^{-1})^\kappa{}_\lambda T^{\rho\sigma}{}_\kappa(\Lambda^{-1}x') \tag{440}$$

8.5 Lorentz covariance of Maxwell's equations

Because tensors transform homogeneously under Lorentz transformations, a set of dynamical equations that set a tensor to zero will automatically retain its form in all inertial frames. Thus we seek to cast Maxwell's equations in this way. The first step is to identify the electric and magnetic fields as the components of some tensor. We start by expressing them in terms of potentials

$$E_k = -\nabla_k\phi - \frac{\partial A_k}{\partial t} = \partial_k(-\phi) - c\partial_0 A_k \equiv c\partial_k A_0 - c\partial_0 A_k \tag{441}$$

$$\epsilon_{ijk}B^k = \epsilon_{ijk}\epsilon^{kmn}\nabla_m A_n = \partial_i A_j - \partial_j A_i \tag{442}$$

We already know that ∂_μ transforms as a covariant 4-vector. So the above forms strongly suggest that we should interpret $A_\mu = (-\phi/c, A_k)$ also as a covariant four vector and $F_{\mu\nu} = c(\partial_\mu A_\nu - \partial_\nu A_\mu)$ as a second rank covariant 4-tensor. Since it is antisymmetric, there are precisely 6 independent components, corresponding to the 6 components of $\boldsymbol{E}$ and $\boldsymbol{B}$. We may write out the components of $F_{\mu\nu}$ as a 4×4 matrix array as follows:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & cB^3 & -cB^2 \\ E^2 & -cB^3 & 0 & cB^1 \\ E^3 & cB^2 & -cB^1 & 0 \end{pmatrix} \tag{443}$$

Let us work out what the tensor transformation law $F'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}$ implies for the electric and magnetic fields. Consider boosts in the x direction, so $\Lambda^0{}_0 = \Lambda^1{}_1 = \gamma$, $\Lambda^0{}_1 = \Lambda^1{}_0 = -\gamma v/c$, and $\Lambda^2{}_2 = \Lambda^3{}_3 = 1$. Then

$$E'^1 = F'_{10} = F'^{01} = \Lambda^0{}_0\Lambda^1{}_1 F^{01} + \Lambda^1{}_0\Lambda^0{}_1 F^{10} = F^{01} = E^1 \tag{444}$$

$$cB'^1 = F'_{23} = F'^{23} = \Lambda^2{}_2\Lambda^3{}_3 F^{23} = F^{23} = cB^1 \tag{445}$$

$$\begin{aligned} E'^2 &= F'^{02} = \Lambda^2{}_2(\Lambda^0{}_0 F^{02} + \Lambda^0{}_1 F^{12}) \\ &= \gamma\left(E^2 - vB^3\right) = \gamma\left(E^2 + (\boldsymbol{v}\times\boldsymbol{B})^2\right) \end{aligned} \tag{446}$$

$$\begin{aligned} cB'^2 &= F'^{31} = \Lambda^3{}_3(\Lambda^1{}_1 F^{31} + \Lambda^1{}_0 F^{30}) \\ &= \gamma\left(cB^2 + \frac{v}{c}E^3\right) = \gamma\left(cB^2 - (\frac{\boldsymbol{v}}{c}\times\boldsymbol{E})^2\right) \end{aligned} \tag{447}$$

$$E'^3 = \gamma\left(E^3 + (\boldsymbol{v}\times\boldsymbol{B})^3\right), \qquad cB'^3 = \gamma\left(cB^3 - (\frac{\boldsymbol{v}}{c}\times\boldsymbol{E})^3\right) \tag{448}$$

We can infer from these formulas the boosts in a general direction $\boldsymbol{v}$, by defining $_\parallel$ and $_\perp$ as the components of a vector parallel to and perpendicular to $\boldsymbol{v}$ respectively. Then

$$\boldsymbol{E}'_\parallel = \boldsymbol{E}_\parallel, \qquad \boldsymbol{B}'_\parallel = \boldsymbol{B}_\parallel \tag{449}$$

$$\boldsymbol{E}'_\perp = \gamma(\boldsymbol{E}_\perp + \boldsymbol{v}\times\boldsymbol{B}), \qquad \boldsymbol{B}'_\perp = \gamma\left(\boldsymbol{B}_\perp - \frac{\boldsymbol{v}}{c^2}\times\boldsymbol{E}\right) \tag{450}$$

Since we have used potentials, the homogeneous Maxwell equations are automatically satisfied. The remaining ones are

$$\rho = \epsilon_0\nabla\cdot\boldsymbol{E} = \epsilon_0\partial_k F_{k0} = -\epsilon_0\partial_k F^{k0} = \epsilon_0\partial_k F^{0k} \tag{451}$$

$$\begin{aligned} J^i &= \frac{1}{\mu_0}\epsilon_{ikm}\partial_k B^m - c\epsilon_0\partial_0 E^i = \frac{1}{c\mu_0}\partial_k F_{ik} - c\epsilon_0\partial_0 F_{i0} \\ &= c\epsilon_0\partial_k F^{ik} + c\epsilon_0\partial_0 F^{i0} \end{aligned} \tag{452}$$

Thus we see that if we interpret $J^\mu = (c\rho, J^k)$ as a contravariant 4-vector, Maxwell's equations assume the form[5]

$$\epsilon_0\partial_\nu F^{\mu\nu} = \frac{1}{c}J^\mu \equiv \rho^\mu, \tag{453}$$

[5]In SI units electric and magnetic fields have different units. Since we have defined $F_{\mu\nu}$ to have the units of the electric field, $\epsilon_0\partial_\nu F^{\mu\nu}$ has dimensions of charge density as does $J^\mu/c \equiv \rho^\mu$. Perhaps we should have defined a field strength $B_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}/c$ with dimensions of the magnetic field. Then Maxwell's equations would assume the form $\epsilon_0 c^2\partial_\nu B^{\mu\nu} = \partial_\nu B^{\mu\nu}/\mu_0 = J^\mu$, which resembles the magnetic Ampère-Maxwell equation.

which resembles the Gauss-Maxwell equation. This puts the equations in manifestly Lorentz covariant form, so they take this form in all inertial frames connected by Lorentz transformations. Note that charge conservation $0 = \partial_\mu J^\mu = \nabla \cdot \boldsymbol{J} + \frac{\partial\rho}{\partial t}$ is a direct consequence of the antisymmetry of $F^{\mu\nu} = -F^{\nu\mu}$. Its validity in all inertial frames follows from the 4-vector interpretation of J^μ.

Although we have used potentials to identify the fields as components of $F_{\mu\nu}$, we can dispense with them by restoring the homogeneous Maxwell equations. We cast them in covariant form as follows:

$$\begin{aligned} 0 &= \nabla \cdot \boldsymbol{B} = \frac{1}{c}(\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12}) \\ 0 &= (\nabla \times \boldsymbol{E})^i + \frac{\partial \boldsymbol{B}^i}{\partial t} = \epsilon^{ijk}\partial_j F_{k0} + \frac{1}{2}\partial_0 \epsilon^{ijk} F_{jk} \\ &= \frac{1}{2}\epsilon^{ijk}(\partial_j F_{k0} + \partial_k F_{0j} + \partial_0 F_{jk}) \end{aligned} \tag{454}$$

These 4 equations are the various components of the covariant condition

$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0 \tag{455}$$

One can check directly that the left side is antisymmetric under the interchange of any pair of indices, which means that no information is lost if one multiplies by the completely antisymmetric 4-tensor $\epsilon^{\kappa\mu\nu\rho}$ and sums over repeated indices:

$$0 = \epsilon^{\kappa\mu\nu\rho}\partial_\mu F_{\nu\rho} = \partial_\mu(\epsilon^{\kappa\mu\nu\rho}F_{\nu\rho}) \equiv 2\partial_\mu F^{*\kappa\mu} \tag{456}$$

where $F^{*\kappa\mu} \equiv \epsilon^{\kappa\mu\nu\rho}F_{\nu\rho}/2$ is known as the *dual* of $F_{\mu\nu}$. This dual transformation interchanges the roles of electric and magnetic fields.

Incidentally, from the definition of determinants $\Lambda^\kappa{}_\rho \Lambda^\lambda{}_\sigma \Lambda^\mu{}_\tau \Lambda^\nu{}_\omega \epsilon^{\rho\sigma\tau\omega} = \epsilon^{\kappa\lambda\mu\nu} \det\Lambda$, we see that ϵ is invariant under Lorentz transformations (actually under all linear transformations!) with unit determinant. The defining condition of Lorentz transformations shows that transforming $\eta_{\mu\nu}$ as a covariant tensor leaves it unchanged. Thus we can freely use both $\eta_{\mu\nu}$ and $\epsilon^{\kappa\lambda\mu\nu}$ in the process of constructing new tensors (or pseudotensors) from old ones. Some examples:

$$\eta^{\mu\rho}\eta^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} = F_{\mu\nu}F^{\mu\nu} = 2(c^2\boldsymbol{B}^2 - \boldsymbol{E}^2) \tag{457}$$

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = F_{\mu\nu}F^{*\mu\nu} = -4c\boldsymbol{E}\cdot\boldsymbol{B} \tag{458}$$

are two important scalar fields. The first times $-\epsilon_0/4$ is just the Lagrangian density for the electromagnetic field.

$$\begin{aligned} S &= \frac{\epsilon_0}{2}\int dt d^3x(\boldsymbol{E}^2 - c^2\boldsymbol{B}^2) = \frac{1}{2\mu_0}\int dt d^3x(\boldsymbol{E}^2/c^2 - \boldsymbol{B}^2) \\ &= -\frac{\epsilon_0}{4}\int dt d^3x \eta^{\mu\rho}\eta^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} = -\frac{\epsilon_0}{4}\int dt d^3x F_{\mu\nu}F^{\mu\nu} \\ &= -\frac{\epsilon_0}{4c}\int d^4x F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4c\mu_0}\int d^4x B_{\mu\nu}B^{\mu\nu} \end{aligned} \tag{459}$$

where we understand $d^4x = d(ct)d^3x$. These are the only two invariants quadratic in the fields.

Finally we need to confirm that the equations of motion for a charged particle in electric and magnetic fields are the same in all inertial frames. Before we worry about a particle moving in fields we should say a word about the motion of free particles. Then Newton's equation just says that the three momentum $\boldsymbol{p}$ of a free particle is a constant. Then the energy $E = \sqrt{\boldsymbol{p}^2c^2 + m^2c^4}$ is also a constant and we have the Lorentz covariant statement that p^μ is constant. More generally, there will be conservation of the total energy and momentum of a system made up of several interacting particles. For example a scattering process in which two particles of momenta p_1^μ, p_2^μ collide and emerge with altered momenta $p_1'^\mu, p_2'^\mu$, will respect the conservation laws $p_1^\mu + p_2^\mu = p_1'^\mu + p_2'^\mu$ in all inertial systems. In this context one can define Lorentz invariant scalar products like $(p_1 + p_2)\cdot(p_1 + p_2)$ which have the same values in all frames. Two common frames are the Lab frame where $\boldsymbol{p}_2 = 0$ and the center of mass frame where $\boldsymbol{p}_1 + \boldsymbol{p}_2 = 0$. By evaluating the invariant in these two frames we learn that

$$(p_1+p_2)^2 = -\frac{1}{c^2}E_{\rm CM}^2 = p_1^2 + p_2^2 + 2p_1\cdot p_2 = -m_1^2c^2 - m_2^2c^2 - 2m_2E_{1,\rm Lab}$$

$$E_{\rm CM} = c\sqrt{m_1^2c^2 + m_2^2c^2 + 2m_2E_{1,\rm Lab}} \sim c\sqrt{2m_2E_{1,\rm Lab}} \qquad (460)$$

at high energies. To make heavy particles we need high $E_{\rm CM}$ which means much higher $E_{1,\rm Lab}$ because of the square root. This is why colliding beams are much more effective than single beams colliding with stationary targets in producing novel heavy particles.

Turning to the motion of a particle in an electromagnetic field, we seek to cast the Newton equation

$$\frac{d\boldsymbol{p}}{dt} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}), \qquad \boldsymbol{p} = \gamma m\boldsymbol{v} \qquad (461)$$

as a 4-tensor equation. We have already seen that $p^\mu = (E/c, \boldsymbol{p})$ is a contravariant 4-vector. It is convenient to divide by the mass to form the "velocity" 4-vector $U^\mu = (\gamma c, \gamma\boldsymbol{v})$. Then it is natural to consider the quantity

$$F_{\mu\nu}U^\nu = \gamma(cF_{\mu0} + F_{\mu k}v^k) \qquad (462)$$

$$F_{j\nu}U^\nu = \gamma(cE^j + c\epsilon_{jkl}v^kB^l) = c\gamma(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})^j \qquad (463)$$

$$F_{0\nu}U^\nu = \gamma F_{0k}v^k = -\gamma\boldsymbol{v}\cdot\boldsymbol{E} \qquad (464)$$

Then Newton's equation can be written

$$\frac{dp^j}{dt} = \frac{q}{\gamma c}F_{j\nu}U^\nu \qquad (465)$$

which are the spatial components of the 4-vector equation

$$\gamma\frac{dp_\mu}{dt} = \frac{q}{c}F_{\mu\nu}U^\nu \qquad (466)$$

But this equation has a time component which seems independent of the Newton equation:

$$\frac{dp_0}{dt} = -\frac{1}{c}\frac{dE}{dt} = -\frac{q}{c}\boldsymbol{v}\cdot\boldsymbol{E}, \qquad \frac{d}{dt}\sqrt{\boldsymbol{p}^2c^2+m^2c^4} = q\boldsymbol{v}\cdot\boldsymbol{E} \tag{467}$$

which, since $q\boldsymbol{v}\cdot\boldsymbol{E}$ is just the rate at which work is done on the particle by the electric field, we recognize as the relativistic work energy theorem. It is an easy consequence of the Newton equation:

$$\frac{d}{dt}\sqrt{\boldsymbol{p}^2c^2+m^2c^4} = \frac{\boldsymbol{p}c^2}{\gamma mc^2}\cdot q(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}) = q\boldsymbol{v}\cdot\boldsymbol{E} \tag{468}$$

and so is not really an independent equation. Finally, we note that $dt/\gamma = dt\sqrt{1-\boldsymbol{v}^2/c^2} \equiv d\tau$ is just the time interval dt as measured by a clock in the instantaneous particle rest frame. The measure of time given by τ is therefore a Lorentz invariant notion and τ is accordingly known as the particle's proper time. With this concept we can write Newton's equation in manifestly Lorentz covariant form:

$$\frac{dp_\mu}{d\tau} = m\frac{dU^\mu}{d\tau} = \frac{q}{c}F_{\mu\nu}U^\nu \tag{469}$$

Here Lorentz covariance implicitly requires that the charge q has the same value for all inertial observers. Since it is a property of the particle we could simply assume it. However, the statement should be consistent with the 4-vector transformation properties of J^μ. For this we need the local statement of charge conservation: $\partial_\mu J^\mu = 0$, from which we can write

$$0 = \int d^4x\partial_\mu J^\mu = \oint d^3Sn_\mu J^\mu \tag{470}$$

where on the right we have used Gauss' theorem to write the integral of a divergence as a boundary integral. Now we assume all components of J^μ vanish rapidly enough at *spatial* infinity, so that any part of the boundary at spatial infinity can be dropped. We may freely choose the spatially finite parts of the closed contour. Consider the region bounded by two hypersurfaces, one at $t = t_1$ and the other at $t' = t_2'$, where t, t' are times in two different inertial frames. Then the relation becomes

$$0 = \int d^3xJ^0(\boldsymbol{x}, t_1) - \int d^3x'J'^0(\boldsymbol{x}', t_2') = Q(t_1) - Q'(t_2') \tag{471}$$

This equation can be read in two ways. First, if the primed observer is in fact the same as the unprimed observer, it reads $Q(t_2) = Q(t_1)$, i.e. charge is conserved in all inertial frames. Then if we take the case that the primed observer is boosted, we learn that $Q' = Q$, i.e. charge is a scalar and has the same value at all times in all inertial frames.

This completes the demonstration that Maxwell's electrodynamics takes the same form in all inertial frames connected by Lorentz transformations. This statement is concisely summarized by:

$$\epsilon_0\partial_\nu F^{\mu\nu} = \frac{1}{c}J^\mu, \qquad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0, \qquad \frac{dp_\mu}{d\tau} = \frac{q}{c}F_{\mu\nu}U^\nu = \frac{q}{mc}F_{\mu\nu}p^\nu \tag{472}$$

We should emphasize, that apart from justifying the choice of any convenient inertial frame for the solution of a given problem, the covariant formalism does not really offer new techniques for solving problems. For that one simply settles on a convenient frame and solves the equations as we did last semester.

8.6 Action Principles

We have already mentioned the action for electromagnetic fields coupled to sources which can be put in covariant form as follows[6]:

$$\begin{aligned} S_{fields} &= \int dt d^3x \left[\frac{\epsilon_0}{2}(\boldsymbol{E}^2 - c^2\boldsymbol{B}^2) - \rho\phi + \boldsymbol{J}\cdot\boldsymbol{A}\right] \\ &= \frac{1}{c}\int d^4x \left[-\frac{\epsilon_0}{4}F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu\right] \end{aligned} \tag{473}$$

Maxwell's equations follow from $\delta S = 0$ under variations of the vector potential δA_μ, so $\delta F_{\mu\nu} = c(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)$:

$$\begin{aligned} \delta S &= \frac{1}{c}\int d^4x \left[-\frac{\epsilon_0}{2}F^{\mu\nu}\delta F_{\mu\nu} + J^\mu \delta A_\mu\right] = \frac{1}{c}\int d^4x \left[-\epsilon_0 F^{\mu\nu}\partial_\mu \delta c A_\nu + J^\mu \delta A_\mu\right] \\ &= \frac{1}{c}\int d^4x \left[-c\partial_\mu(\epsilon_0 F^{\mu\nu}\delta A_\nu) + (J^\nu + \epsilon_0 \partial_\mu c F^{\mu\nu})\delta A_\nu\right] \end{aligned} \tag{474}$$

Taking $\delta A_\mu = 0$ at the space-time boundaries, but otherwise arbitrary, then $\delta S = 0$ implies Maxwell's equations, $\epsilon_0 \partial_\nu F^{\mu\nu} = J^\mu / c$.

The relativistic version of the action for a particle moving in fixed fields can also be put in covariant form:

$$\begin{aligned} S_{part} &= -mc^2 \int dt \sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}} - q\int dt(\phi(\boldsymbol{r}(t), t) - \boldsymbol{v}\cdot\boldsymbol{A}(\boldsymbol{r}(t)), t) \\ &= -mc^2 \int d\tau + q\int d\tau U^\mu A_\mu \end{aligned} \tag{475}$$

The last form shows that the action is a scalar, assuming the same form in all inertial reference frames. This is another way of understanding why Newton's equation is covariant.

The complete Maxwell electrodynamics follows from the combined action

$$S = -\frac{\epsilon_0}{4c}\int d^4x F_{\mu\nu}F^{\mu\nu} - mc^2\int d\tau + q\int d\tau U^\mu A_\mu \tag{476}$$

[6]One could also add a term $\theta \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ to the action. This term is invariant under proper Lorentz transformations but it is odd under parity and also odd under time reversal. If we want to keep parity or time reversal symmetry, it would be forbidden. But even if it is allowed, it has no effect on the classical equations of motion because one can write $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = 2\partial_\mu(\epsilon^{\mu\nu\rho\sigma} A_\nu F_{\rho\sigma})$ showing that its variation will not contribute to δS under the conditions of Hamilton's principle.

which is also a Lorentz scalar. It therefore implies covariant equations of motion, which of course we have already shown directly. The current due to the particle is, by equating the last term to $(1/c)\int d^4xJ^\mu A_\mu$, yielding $J^\mu = qc\int d\tau U^\mu\delta^4(x-x(\tau))$. Explicitly, we have

$$\begin{aligned}
\rho c &= J^0 = qc^2\int d\tau\gamma\delta(ct-x^0(\tau))\delta(\boldsymbol{x}-\boldsymbol{x}(\tau)) = qc^2\frac{\gamma}{dx^0/d\tau}\delta(\boldsymbol{x}-\boldsymbol{x}(\tau(t))) \\
\rho &= q\delta(\boldsymbol{x}-\boldsymbol{x}(\tau(t))) \qquad (477) \\
\boldsymbol{J} &= qc\int d\tau\gamma\frac{d\boldsymbol{x}}{dt}\delta(ct-x^0(\tau))\delta(\boldsymbol{x}-\boldsymbol{x}(\tau)) = q\frac{d\boldsymbol{x}}{dt}\delta(\boldsymbol{x}-\boldsymbol{x}(\tau)) \qquad (478)
\end{aligned}$$

where $\tau(t)$ is implicitly defined by $ct = x^0(\tau(t))$. These are the familiar equations we used last semester.

8.7 Some particle motions in electromagnetic fields

The motion of a charged particle in uniform fields $F_{\mu\nu}(x)$ =constant is an important case. The covariant form of the equations of motion

$$\frac{dU^\mu}{d\tau} = \frac{q}{mc}F^\mu{}_\nu U^\nu, \qquad U^\mu = \frac{dx^\mu}{d\tau}, \qquad U_\mu U^\mu = -c^2 \qquad (479)$$

reduces to a set of 4 coupled first order differential equations for $U^\mu(\tau)$ with constant coefficients. If we employ matrix notation, the solution can be expressed as $U(\tau) = e^{qF\tau/mc}U(0)$. This formal solution can be made more concrete by expressing it as a linear combination of 4 "normal mode" solutions in each of which all components of U have a common exponential τ dependence $U^\mu = V_r^\mu e^{qr\tau/mc}$. Then one needs to solve a matrix eigenvalue problem $FV_r = rV_r$, with the matrix elements of F given by $F^\mu{}_\nu$. This is a straightforward problem in linear algebra. One determines the possible eigenvalues r from the characteristic equation $\det(F-rI) = 0$. The Lorentz transformation rules

$$F'^\mu{}_\nu = \Lambda^\mu{}_\rho F^\rho{}_\sigma(\Lambda^{-1})^\sigma{}_\nu = (\Lambda F\Lambda^{-1})^\mu{}_\nu \qquad (480)$$

Show that $\det(F'-rI) = \det(F-rI) = 0$, so the characteristic equation may be evaluated in any Lorentz frame, for example, the one in which the electric and magnetic fields are both parallel to the x-axis. In that frame

$$F = \begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & cB \\ 0 & 0 & -cB & 0 \end{pmatrix}, \qquad \det(F-rI) = (r^2-E^2)(r^2+B^2) \qquad (481)$$

so the four possible eigenvalues are $r = \pm E$ and $r = \pm icB$. The general solution will then be $U^\mu = \sum_r c_r V_r^\mu e^{qr\tau/mc}$. The constraint $U_\mu U^\mu = -c^2$ will be independent of τ because $\eta_{\mu\nu}V_r^\mu V_{r'}^\nu = 0$ unless $r + r' = 0$. A simple further integration yields

$$x^\mu(\tau) = x^\mu(0) + \sum_r \frac{c_r mc}{qr}V_r^\mu(e^{qr\tau/mc}-1) \equiv X^\mu + \sum_r \frac{c_r mc}{qr}V_r^\mu e^{qr\tau/mc} \qquad (482)$$

To represent the trajectory in space as a function of the frame's time one would have to invert $ct = x^0(\tau)$ to get $\tau(t)$ and then obtain $\boldsymbol{x}(\tau(t))$. In the general case this last step is complicated.

A general feature of the motion can be brought out by noticing that

$$
\begin{aligned}
(x(\tau)-X)^2 &= \sum_{r,s} \frac{c_r c_s m^2 c^2}{q^2 rs} V_r \cdot V_s \; e^{q(r+s)\tau/mc} \\
(x(\tau)-X)^2 &= -\sum_{r} \frac{c_r c_{-r} m^2 c^2}{q^2 r^2} V_r \cdot V_{-r} \qquad (483)
\end{aligned}
$$

which is to say that $x^\mu(\tau)$ lies on a hyperboloid. We next find the motion in a few special situations.

Uniform electric field The equation of motion can be immediately integrated to yield $\boldsymbol{p}(t) = \boldsymbol{p}_0 + q\boldsymbol{E}(t-t_0)$, where without loss of generality we can take $\boldsymbol{p}_0 \cdot \boldsymbol{E} = 0$ and $t_0 = 0$. Then since $\boldsymbol{v} = d\boldsymbol{r}/dt = \boldsymbol{p}c/\sqrt{\boldsymbol{p}^2 + m^2c^2}$, we have

$$
\begin{aligned}
\boldsymbol{r}(t) &= \boldsymbol{r}_0 + \int_0^t dt' \frac{\boldsymbol{p}_0 c + qc\boldsymbol{E}t'}{\sqrt{q^2\boldsymbol{E}^2 t'^2 + \boldsymbol{p}_0^2 + m^2c^2}} \\
&\sim ct\hat{\boldsymbol{E}} + \frac{\boldsymbol{p}_0 c}{q|\boldsymbol{E}|} \ln t + O(1) \qquad (484)
\end{aligned}
$$

which shows that at large times the particle's speed approaches that of light.

Boosted uniform electric field A boost parallel to $\boldsymbol{E}$ leaves the field invariant and simply maps the solution on to the analogous solution in the primed frame. To see this, take $\boldsymbol{E} = E\hat{x}$ and $\boldsymbol{p}_0 = p_0\hat{y}$. Then taking a boost in the x-direction the transformed momentum is

$$
p'_x = \gamma\left(qEt - \frac{v}{c}\sqrt{q^2\boldsymbol{E}^2t^2 + \boldsymbol{p}_0^2 + m^2c^2}\right), \qquad p'_y = p_0, \qquad p'_z = 0 \qquad (485)
$$

The time in the new frame $t' = \gamma(t - vx(t)/c^2)$ can be obtained by computing

$$
\begin{aligned}
x(t) &= x_0 + \frac{c}{qE}\left(\sqrt{q^2\boldsymbol{E}^2t^2 + \boldsymbol{p}_0^2 + m^2c^2} - \sqrt{\boldsymbol{p}_0^2 + m^2c^2}\right) \qquad (486) \\
t' &= \gamma\left(t - \frac{vx_0}{c^2} - \frac{v}{qcE}(\sqrt{q^2\boldsymbol{E}^2t^2 + \boldsymbol{p}_0^2 + m^2c^2} - \sqrt{\boldsymbol{p}_0^2 + m^2c^2})\right) \qquad (487) \\
\boldsymbol{p}' &= \boldsymbol{p}_0 + q\boldsymbol{E}t' + \text{const} \qquad (488)
\end{aligned}
$$

A boost perpendicular to $\boldsymbol{E}$ generates a magnetic field in the primed frame, ${\boldsymbol{E}'}_\perp = \gamma\boldsymbol{E}_\perp$ and $\boldsymbol{B}'_\perp = -\boldsymbol{v} \times \boldsymbol{E}'_\perp/c^2$. This is a new physical situation: we can express $v = c^2B'/E'$ and $\gamma = 1/\sqrt{1 - c^2B'^2/E'^2}$. Consider the above solution in the special case $\boldsymbol{p}_0 = 0$, with $\boldsymbol{E} = E\hat{y}$ and putting $x_0 = z_0 = 0$,

$$
\begin{aligned}
x(t) &= z(t) = 0 \\
y(t) &= y_0 + \int_0^t dt' \frac{qcEt'}{\sqrt{q^2\boldsymbol{E}^2t'^2 + m^2c^2}} = Y + c\hat{y}\sqrt{t^2 + m^2c^2/q^2E^2} \qquad (489)
\end{aligned}
$$

in a frame moving with velocity $v\hat{x}$ in the x-direction, so $\boldsymbol{E}' = \gamma E\hat{y}$ and $\boldsymbol{B}' = -\gamma vE\hat{z}/c^2 = -vE'\hat{z}/c^2$.

$$\begin{aligned}
t' &= \gamma(t - vx(t)/c^2) = \gamma t, \qquad x' = \gamma(x(t) - vt) = -vt' = -\frac{c^2B'}{E'}t', \qquad z' = 0 \\
y'(t') &= y(t) = y(t'/\gamma) = Y + c\sqrt{\frac{t'^2}{\gamma^2} + \frac{m^2c^2\gamma^2}{q^2E'^2}} \\
&= Y + c\sqrt{t'^2\left(1 - \frac{c^2B'^2}{E'^2}\right) + \frac{m^2c^2}{q^2(E'^2 - c^2B'^2)}}
\end{aligned} \tag{490}$$

This gives us the solution for motion in perpendicular electric and magnetic fields. There is accelerated motion in the direction of $\boldsymbol{E}$ and a uniform drift at speed c^2B'/E' in the direction of $\boldsymbol{E}' \times \boldsymbol{B}'$.

Uniform magnetic field

$$\frac{d\boldsymbol{p}}{dt} = q\boldsymbol{v} \times \boldsymbol{B} \tag{491}$$

Dotting both sides with $\boldsymbol{p}$ shows that $\boldsymbol{p}^2$ and hence $\boldsymbol{v}^2$ and γ are constants. Thus we can write the equation of motion as $d\boldsymbol{v}/dt = \boldsymbol{v} \times \boldsymbol{\omega}$, where $\boldsymbol{\omega} = q\boldsymbol{B}/m\gamma$. Thus $\boldsymbol{v}_\parallel$ is constant and $\boldsymbol{v}_\perp$ rotates in the transverse plane with angular frequency $|\boldsymbol{\omega}|$. Explicitly, with $\boldsymbol{B} = B\hat{z}$, $v_z =$ constant, $v_x = v_\perp \cos\omega t$, $v_y = v_\perp \sin\omega t$, and

$$z = v_z t, \qquad x = x_0 + \frac{v_\perp}{\omega}\sin\omega t, \qquad y = y_0 - \frac{v_\perp}{\omega}\cos\omega t \tag{492}$$

Thus the particle moves in a circle of radius $R_B = v_\perp/\omega = mv_\perp\gamma/qB = p_\perp/qB$.

Boosted uniform magnetic field Again a boost parallel to the field just scales the field without altering the physical situation. So we just examine a boost perpendicular to the field, which we take in the z-direction $\boldsymbol{B} = B\hat{z}$. So our canonical boost in the x-direction $\boldsymbol{v} = v\hat{x}$ does what we want. In the primed frame $\boldsymbol{B}' = \gamma B\hat{z}$ And an electric field $\boldsymbol{E}' = E'\hat{y} = \gamma\boldsymbol{v} \times \boldsymbol{B} = \boldsymbol{v} \times \boldsymbol{B}' = -vB'\hat{y}$ appears. So we have perpendicular electric and magnetic fields with $|E'| < c|B'|$. To find the motion in such fields all we have to do is boost the solution we found with $\boldsymbol{E} = 0$.

For simplicity let's just consider the circular motion in the xy-plane:

$$x(t) = \frac{v_0}{\omega_B}\cos\omega_B t, \qquad y(t) = -\frac{v_0}{\omega_B}\sin\omega_B t, \qquad z(t) = 0 \tag{493}$$

The boost is then

$$x' = \gamma\left(\frac{v_0}{\omega_B}\cos\omega_B t - vt\right), \quad t' = \gamma\left(t - \frac{vv_0}{c^2\omega_B}\cos\omega_B t\right), \quad y' = -\frac{v_0}{\omega_B}\sin\omega_B t, \quad z' = 0$$

We see from the x' equation that there is an average drift in the negative x'-direction. To express the boosted solution in primed frame variables, we note that $\boldsymbol{v} = -\boldsymbol{E}' \times \boldsymbol{B}'/\boldsymbol{B}'^2$

so the average drift $-\boldsymbol{v}$ of the gyrating particle is in the direction of $\boldsymbol{E}'\times\boldsymbol{B}'$. Also $\omega_B = qB/m\gamma_0 = qB'/m\gamma_0\gamma$, where $\gamma_0 = 1/\sqrt{1-v_0^2/c^2}$. Ideally we would eliminate t in favor of t', but we can only do that implicitly. We can do a partial elimination however

$$x' = -vt' + \gamma(1-v^2/c^2)\frac{v_0}{\omega_B}\cos\omega_B t = -vt' + \frac{v_0}{\omega_B\gamma}\cos\omega_B t \tag{494}$$

A simple statement can be made at the times when $\cos\omega_B t = 0$, i.e. the times when $x' = -vt'$, and $y' = \pm v_0/\omega_B \equiv \pm Y_0$, its extreme values. These are at $t' = \gamma t = \gamma(n+1/2)\pi/\omega_B$.

The special case $v_0 = 0$ is interesting. The particle is at rest in the unprimed frame, where $\boldsymbol{E} = 0$. In the primed frame it moves uniformly at velocity $-\boldsymbol{v}$. This is consistent with the equations of motion since $\boldsymbol{E}' = \boldsymbol{v}\times\boldsymbol{B}'$ so that $\boldsymbol{F} = q(\boldsymbol{E}' - \boldsymbol{v}\times\boldsymbol{B}') = 0$. The magnetic force cancels the electric force. This means that perpendicular electric and magnetic fields can be used as a velocity filter: only particles with velocity $\boldsymbol{V} = \boldsymbol{E}'\times\boldsymbol{B}'/\boldsymbol{B}'^2$ will pass through undeflected.

Drift in Inhomogeneous Magnetic Fields. It is not generally possible to find analytic solutions for nonuniform magnetic fields. But if the fields vary slowly one may use the solution in uniform fields as the starting point for an approximate solution. First consider the case where the magnetic field is in the z direction, but is consistently stronger in one transverse direction, say in the x direction: $\boldsymbol{B} \approx \hat{z}(B_0 + xB')$. if we ignore the B' term, a simple motion would follow a circle in the xy plane at constant speed. Including the xB' term would tighten the circle for $x > 0$ but loosen the circle for $x < 0$. Thus there should be a systematic drift parallel to the y-axis. To estimate this drift velocity, we write $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1$, where $\boldsymbol{v}_0$ is the solution for uniform field. Then

$$\frac{d\boldsymbol{v}_1}{dt} \approx \frac{q}{m}[xB'\boldsymbol{v}_0\times\hat{z} + \boldsymbol{v}_1\times B_0\hat{z}] + O(v_1B') \tag{495}$$

We can identify the drift velocity as $\langle\boldsymbol{v}_1\rangle$, the time average over a cycle of the zeroth order motion. Averaging the above equation gives

$$[B'\langle x\boldsymbol{v}_0\rangle + \langle\boldsymbol{v}_1\rangle B_0]\times\hat{z}] \approx 0 \tag{496}$$

Presuming the drift velocity is in the xy plane, we conclude

$$\boldsymbol{v}_{drift} \approx -\frac{B'}{B_0}\langle x\boldsymbol{v}_0\rangle \tag{497}$$

The average involves the zeroth order solutions

$$\begin{aligned} x(t) &= a\cos\omega_B t, \qquad y = -a\sin\omega_B t, \\ v_0^x &= -a\omega_B\sin\omega_b t, \qquad v_0^y = -a\omega_B\cos\omega_B t \end{aligned} \tag{498}$$

with $\omega_B = qB/m\gamma$. We then see that

$$\boldsymbol{v}_{drift} \approx \frac{B'}{2B_0}a^2\omega_B\hat{y} = -\frac{\nabla_\perp B\times\boldsymbol{B}}{2B^2}a^2\omega_B \tag{499}$$

where we wrote the last form in a way that doesn't depend on the choice of coordinate axes. The sign of ω_B matters: it is the same as the sign of the charge.

Adiabatic Invariants for slowly varying magnetic fields. I will just say a few words about the use of Adiabatic invariants to explain some interesting features of the motion if the gyrating particle has a component of velocity in the direction of the $\boldsymbol{B}$ field.

The canonical formalism of classical mechanics draws attention to the action integrals

$$I_k = \oint p_k dq_k \tag{500}$$

over canonical coordinates that are periodic as the particle follows its trajectory. If some parameter λ in the Hamiltonian varies sufficiently slowly with time, i.e. $\dot{\lambda} \ll \lambda/T$, then even after a long time T during which the Hamiltonian makes a finite change, the values of the I_k will remain unchanged. An example is the helical motion of a particle about a magnetic field line which is periodic in the coordinates perpendicular to the field. The corresponding action integral is

$$I = \oint (\gamma m\boldsymbol{v} + q\boldsymbol{A}) \cdot d\boldsymbol{l} = 2\pi\gamma m\omega_B a^2 + q \int dS \boldsymbol{n} \cdot \boldsymbol{B} \tag{501}$$

where we used Stokes theorem. The normal $\boldsymbol{n}$ is in the direction such that the contour is in a counterclockwise sense looking down on it. The sign of the first integral is written assuming the velocity is in the same sense. This requires that the $\boldsymbol{B}$ is antiparallel to $\boldsymbol{n}$. Thus

$$I = 2\pi\gamma m\omega_B a^2 - q\pi a^2 B = \pi a^2 qB \tag{502}$$

Now consider a particle initially gyrating around a field line with a component of initial velocity parallel to the field in a direction of a slowly increasing field. If the field is static the particle's energy γmc^2 will be constant. The adiabatic invariant I will also be constant, so a will decrease as $1/\sqrt{B}$. Meanwhile the transverse velocity $v_\perp = a\omega_B = aqB/\gamma m$ will increase as $\sqrt{B}$. Thus the parallel velocity will decrease as

$$v_\parallel^2 = v_0^2 - v_\perp^2 = v_0^2 - v_{0\perp}^2 \frac{B}{B_0} \tag{503}$$

If B increases enough $v_\parallel$ will drop to 0 and the particle will be reflected back from regions of strong magnetic field.

8.8 Electrodynamics of a Scalar Field

Up until now we have developed electrodynamics by modeling the currents and charges with point particles. In such a description two very different types of physical entities, particles and fields must interact with each other in a consistent way. While this hybrid approach to electrodynamics can achieve some level of success, it engenders many nasty complications. A more coherent approach is to try describing all entities in terms of fields. This is possible because quantization endows fields with particle-like properties. Indeed the fields of the standard model have the scope to describe all phenomena seen in experiment thus far. These fields include, in addition to electromagnetic fields, nonabelian gauge fields which mediate the weak and strong interactions, fermionic quark and lepton fields, and finally scalar fields which generate mass. We limit our field description of charged matter to scalar fields.

We start with a free scalar field, $\phi(x)$. The only linear field equation, second order in derivatives and consistent with Lorentz invariance, $\phi'(x') = \phi(\Lambda^{-1}x')$, is

$$(-\partial^2 + \mu^2)\phi = \left(\mu^2 - \nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = 0 \tag{504}$$

An action which leads to this field equation is easy to write down:

$$S_\phi = \int dt d^3x \frac{1}{2}(-\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2) \tag{505}$$

$$\delta S_\phi = \int dt d^3x(-\partial_\mu\delta\phi\partial^\mu\phi - \mu^2\phi\delta\phi) = \int dt d^3x \delta\phi(\partial^2 - \mu^2)\phi = 0 \tag{506}$$

We have already considered such a field when we initially motivated Einstein's modification of particle mechanics. The key was to notice that a packet of plane waves which have the frequency wave number relation $\omega^2 = c^2(\boldsymbol{k}^2 + \mu^2)$ have a group velocity $\boldsymbol{v}_g = \boldsymbol{k}/\sqrt{\boldsymbol{k}^2 + \mu^2}$, which with the Planck-Einstein connection $\boldsymbol{p} = \hbar\boldsymbol{k}$, $E = \hbar\omega$, leads to the usual Einstein modification of momentum $\boldsymbol{p} = \gamma m\boldsymbol{v}$ and $E = \sqrt{m^2c^4 + \boldsymbol{p}^2c^2}$. Now we explore the scalar field in its own right.

Before turning to electrodynamics let's review some basics of Hamilton's principle for fields in a more general setting. In a field theory the action $\int dtL$ is a space time integral rather than just a time integral. We generally restrict the integrand, the Lagrangian density, to be a function of the fields ψ_i and their space-time derivatives

$$S = \frac{1}{c}\int d^4x \mathcal{L}(\psi_i, \partial_\mu\psi_i) \tag{507}$$

$$\delta S = \int d^4x \left[\sum_i \delta\psi_i \frac{\partial\mathcal{L}}{\partial\psi_i} + \sum_i \partial_\mu\delta\psi_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\right] = \int d^4x \sum_i \delta\psi_i \left[\frac{\partial\mathcal{L}}{\partial\psi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\right]$$

where we have dropped a surface term. Thus in this general context the field equations following from $\delta S = 0$ are

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)} = \frac{\partial\mathcal{L}}{\partial\psi_i} \tag{508}$$

The analogue of a particle Hamiltonian in field theory is the conserved canonical energy momentum tensor which is constructed as follows

$$T^{\mu\nu} = -\sum_i \partial^\mu\psi_i \frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi_i)} + \eta^{\mu\nu}\mathcal{L} \tag{509}$$

$$
\begin{aligned}
\partial_\nu T^{\mu\nu} &= -\sum_i \left[\partial^\mu\partial_\nu\psi_i \frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi_i)} + \partial^\mu\psi_i\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi_i)}\right] + \partial^\mu\mathcal{L} \\
&= -\sum_i \left[\partial^\mu\partial_\nu\psi_i \frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi_i)} + \partial^\mu\psi_i\frac{\partial\mathcal{L}}{\partial\psi_i}\right] + \partial^\mu\mathcal{L} = -\partial^\mu\mathcal{L} + \partial^\mu\mathcal{L} = 0 \qquad (510)
\end{aligned}
$$

For the simple scalar field theory we introduced above we have

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} &= -\partial^\nu\phi, \qquad \frac{\partial\mathcal{L}}{\partial\phi} = -\mu^2\phi \qquad (511) \\
T^{\mu\nu} &= \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\frac{1}{2}[\partial_\rho\phi\partial^\rho\phi + \mu^2\phi^2] \qquad (512) \\
T^{00} &= \frac{1}{2}[\dot{\phi}^2 + (\nabla\phi)^2 + \mu^2\phi^2 \qquad (513)
\end{aligned}
$$

In this case $T^{\mu\nu}$ is symmetric in its indices, but the construction doesn't guarantee that.

Another example is the free electromagnetic field, with $\mathcal{L} = -\epsilon_0 F_{\mu\nu}F^{\mu\nu}/4 = -\epsilon_0 c^2\partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu)/2$. Then

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\rho)} &= -\epsilon_0 c F^{\nu\rho}, \qquad \frac{\partial\mathcal{L}}{\partial A_\rho} = 0 \\
T^{\mu\nu} &= \epsilon_0 c\partial^\mu A_\rho F^{\nu\rho} + \eta^{\mu\nu}\mathcal{L} = \epsilon_0 F^\mu{}_\rho F^{\nu\rho} + \eta^{\mu\nu}\mathcal{L} + \epsilon_0 c\partial_\rho A^\mu F^{\nu\rho} \\
&= \epsilon_0 F^\mu{}_\rho F^{\nu\rho} + \eta^{\mu\nu}\mathcal{L} + \epsilon_0 c\partial_\rho(A^\mu F^{\nu\rho}) - \epsilon_0 c A^\mu\partial_\rho F^{\nu\rho} \qquad (514) \\
&\to \epsilon_0 F^\mu{}_\rho F^{\nu\rho} + \eta^{\mu\nu}\mathcal{L} + \epsilon_0 c\partial_\rho(A^\mu F^{\nu\rho}) \qquad (515)
\end{aligned}
$$

because of the equations of motion. The first two terms in the final form are symmetric under $\mu \leftrightarrow \nu$. The last term satisfies $\partial_\nu\epsilon_0 c\partial_\rho(A^\mu F^{\nu\rho}) = 0$ by virtue of the antisymmetry of F. It follows that the first two terms are conserved by themselves. Furthermore the contribution of the last term to the total energy-momentum vanishes:

$$
\int d^3x\epsilon_0 c\partial_\rho(A^\mu F^{0\rho}) = \int d^3x\epsilon_0 c\partial_k(A^\mu F^{0k}) = \epsilon_0 c\oint dS\boldsymbol{n}\cdot(A^\mu\boldsymbol{E}) \to 0 \qquad (516)
$$

for localized fields. Therefore the improved energy momentum tensor $T_\mathrm{S}^{\mu\nu} = \epsilon_0 F^\mu{}_\rho F^{\nu\rho} + \eta^{\mu\nu}\mathcal{L}$ is just as good as the canonical one: indeed it is better since it is both gauge invariant and symmetric. It is also traceless (in 4 dimensions):

$$
\begin{aligned}
T_\mathrm{S}^{\mu\nu} &= \epsilon_0 F^\mu{}_\rho F^{\nu\rho} + \eta^{\mu\nu}\mathcal{L} \qquad (517) \\
T^\mu_{S\mu} &= \epsilon_0 F_{\mu\rho}F^{\mu\rho} + 4\mathcal{L} = 0 \qquad (518)
\end{aligned}
$$

The reason that symmetric is better is that the simple intuitive construction of the angular momentum/boost tensor

$$
\mathcal{M}^{\mu\nu\rho} = x^\mu T_\mathrm{S}^{\nu\rho} - x^\nu T_\mathrm{S}^{\mu\rho} \qquad (519)
$$

is automatically conserved $\partial_\rho \mathcal{M}^{\mu\nu\rho} = T_{\rm S}^{\nu\mu} - T_{\rm S}^{\mu\nu} = 0$. If the canonical one were used one would need to discover an extra term to make it conserved. The densities of energy and momentum computed from T_S are the familiar ones

$$T^{00} = \epsilon_0 \boldsymbol{E}^2 - \frac{\epsilon_0}{2}(\boldsymbol{E}^2 - c^2\boldsymbol{B}^2) = \frac{\epsilon_0}{2}\boldsymbol{E}^2 + \frac{\epsilon_0 c^2}{2}\boldsymbol{B}^2 = \frac{\epsilon_0}{2}\boldsymbol{E}^2 + \frac{1}{2\mu_0}\boldsymbol{B}^2 \tag{520}$$

$$T^{i0} = \epsilon_0 F^i{}_\rho F^{0\rho} = \epsilon_0 F^i{}_k F^{0k} = \epsilon_0 c\epsilon_{ikl} B^l E_k = \epsilon_0 c(\boldsymbol{E}\times\boldsymbol{B})^i \tag{521}$$

$$= c(\boldsymbol{D}\times\boldsymbol{B})^i = cg^i \tag{522}$$

$$= \frac{1}{c}(\boldsymbol{E}\times\boldsymbol{H})^i = \frac{1}{c}S^i \tag{523}$$

the factors of c just reflecting the fact that T^{i0} has dimensions of energy density, whereas $\boldsymbol{g}$ has dimensions of momentum density and $\boldsymbol{S}$ has dimensions of energy flux.

The next question is how to set up a consistent interaction between a scalar field and the electromagnetic fields. For starters we need to find a current J^ν built out of the scalar field which is conserved. $\partial_\nu J^\nu = 0$. With a single real field one possibility would be $\partial_\nu\phi$, but then current conservation would require $\mu = 0$. To have $\mu > 0$ we need to make ϕ complex. Then we can form the combination $J_\nu = -iQ(\phi^*\partial_\nu\phi - \phi\partial_\nu\phi^*)$. Now we check

$$\partial_\nu J^\nu = -iQ(\phi^*\partial^2\phi - \phi\partial^2\phi^*) = -iQ(\mu^2 - \mu^2)|\phi|^2 = 0 \tag{524}$$

by the field equation for ϕ. Consider the combination

$$-\partial_\nu\phi^*\partial^\nu\phi + A_\nu J^\nu = -(\partial_\nu\phi^* + iQA_\nu\phi^*)(\partial^\nu\phi - iQA^\nu\phi) + Q^2 A_\nu A^\nu \phi^*\phi \tag{525}$$

We can make the first term on the right gauge invariant under $A_\nu \to A_\nu + \partial_\nu\Lambda$, provided $\phi \to e^{iQ\Lambda}\phi$, because then

$$(\partial_\nu - iQ(A_\nu + \partial_\nu\Lambda))e^{iQ\Lambda}\phi = e^{iQ\Lambda}(\partial_\nu - iQA_\nu)\phi \tag{526}$$

The last term is not gauge invariant so to make a gauge invariant Lagrangian we should drop it and just keep the first term. Thus the complete Action should be

$$S = \frac{1}{c}\int d^4x \left(-\frac{\epsilon_0}{4}F_{\rho\nu}F^{\rho\nu} - (\partial_\nu + iQA_\nu)\phi^*(\partial^\nu - iQA^\nu)\phi - \mu^2\phi^*\phi\right) \equiv \frac{1}{c}\int d^4x\mathcal{L} \tag{527}$$

Any function of $\phi^*\phi$ can be added to the integrand without disturbing gauge invariance or Lorentz invariance. Thus we could replace $-\mu^2\phi^*\phi$ with $-U(\phi^*\phi)$, so we write more generally

$$\mathcal{L} = -\frac{\epsilon_0}{4}F_{\rho\nu}F^{\rho\nu} - (\partial_\nu + iQA_\nu)\phi^*(\partial^\nu - iQA^\nu)\phi - U(\phi^*\phi) \tag{528}$$

$$= -\frac{\epsilon_0}{4}F_{\rho\nu}F^{\rho\nu} - (D^\mu\phi)^* D_\mu\phi - U(\phi^*\phi) \tag{529}$$

where we introduced the shorthand $D_\mu \equiv \partial^\nu - iQA^\nu$ for the gauge covariant derivative. The dimension of $q = Q\hbar$ is Coulombs. To see this, simply recall from quantum mechanics that $\boldsymbol{p} \leftrightarrow (\hbar/i)\nabla$ and $\boldsymbol{p}$ and $q\boldsymbol{A}$ have the same dimensions.

We now need to rederive the field equations. We need:

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\rho)} &= -\epsilon_0 c F^{\nu\rho}, \qquad \frac{\partial\mathcal{L}}{\partial A_\rho} = -iQ[\phi^*(\partial^\nu - iQA^\nu)\phi - \phi(\partial^\nu + iQA^\nu)\phi^*] \equiv J^\rho \\
\frac{\partial\mathcal{L}}{\partial(\partial_\nu \phi^*)} &= -(\partial^\nu - iQA^\nu)\phi, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\nu \phi)} = -(\partial^\nu + iQA^\nu)\phi^* \qquad (530) \\
\frac{\partial\mathcal{L}}{\partial\phi^*} &= -iQA_\rho(\partial^\rho - iQA^\rho)\phi - \phi U'(\phi^*\phi), \qquad \frac{\partial\mathcal{L}}{\partial\phi} = iQA_\rho(\partial^\rho + iQA^\rho)\phi^* - \phi^* U'(\phi^*\phi)
\end{aligned}
$$

Then the equations of motion are

$$
\begin{aligned}
-\epsilon_0 c\partial_\nu F^{\nu\rho} &= +\epsilon_0 c\partial_\nu F^{\rho\nu} = J^\rho, \qquad J_\rho = -iQ[\phi^*\partial_\rho\phi - \phi\partial_\rho\phi^*] - 2Q^2 A_\rho \phi^*\phi \\
\partial_\nu(\partial^\nu - iQA^\nu)\phi &= iQA_\rho(\partial^\rho - iQA^\rho)\phi + \phi U'(\phi^*\phi), \quad (\partial - iQA)^2\phi = \phi U'(\phi^*\phi) \qquad (531) \\
\partial_\nu(\partial^\nu + iQA^\nu)\phi^* &= -iQA_\rho(\partial^\rho + iQA^\rho)\phi^* + \phi^* U'(\phi^*\phi), \quad (\partial + iQA)^2\phi^* = \phi^* U'(\phi^*\phi)
\end{aligned}
$$

Notice that the last line is just the complex conjugate of the second line. Notice also the very interesting feature that the current J_ρ has a linear term in A_ρ. This arose from our consistent treatment of gauge invariance. It is easy to check that this current is gauge invariant.

Finally for completeness we quote the improved symmetric energy momentum tensor whose derivation is assigned in the homework:

$$
\begin{aligned}
T_S^{\mu\nu} &= \epsilon_0 F^\mu{}_\rho F^{\nu\rho} + (\partial^\mu + iQA^\mu)\phi^*(\partial^\nu - iQA^\nu)\phi + (\partial^\nu + iQA^\nu)\phi^*(\partial^\mu - iQA^\mu)\phi \\
&\quad -\eta^{\mu\nu}\left[\frac{\epsilon_0}{4}F_{\rho\sigma}F^{\rho\sigma} + (\partial_\rho + iQA_\rho)\phi^*(\partial^\rho - iQA^\rho)\phi + U(\phi^*\phi)\right] \qquad (532)
\end{aligned}
$$

In particular the energy density is given by

$$
T_S^{00} = \frac{\epsilon_0}{2}(\boldsymbol{E}^2 + c^2\boldsymbol{B}^2) + (\partial_0 + iQA_0)\phi^*(\partial_0 - iQA_0)\phi + (\nabla + iQ\boldsymbol{A})\phi^* \cdot (\nabla - iQ\boldsymbol{A})\phi + U(\phi^*\phi)
$$

8.9 Lorentz Invariant Superconductivity: The Higgs Mechanism

We have introduced the scalar field into our discussion of electromagnetism with two purposes in mind. The first was to give you a glimpse into the only known systematic way of consistently coupling charged matter to the electromagnetic field. However, to properly understand charged particles in terms of the excitations of a field–scalar or other wise– requires quantum mechanics: the quantum field has particle-like properties, as first realized by Einstein in his explanation of the photoelectric effect in terms of photons. The particles that make up atoms and molecules and indeed all materials must be understood in this way. One starts with quantum field theory, and then imagines a classical limiting description in which the particle aspects of the quantum field theory instead of the classical field/wave aspects dominate the approximation. Then one is back to studying the dynamics of charged point particles interacting with the electromagnetic fields.

We shall use the classical scalar field to understand two related phenomena. The first is the Higgs mechanism, which provides a way to consistently give the photon a mass. Then we will use the insights gained to explain some aspects of superconductivity.

The way a scalar field can generate a mass for the photon can be seen from the behavior of the current for constant nonzero $\phi \to \phi_0$, $J_\rho \to -2Q^2 A_\rho \phi_0^* \phi_0$. In this limit Maxwell's equations read

$$\begin{aligned} \epsilon_0 c \partial_\nu F^{\mu\nu} + 2Q^2 A^\mu \phi_0^* \phi_0 &= \epsilon_0 c^2 \partial_\mu \partial_\nu A^\nu + (-\epsilon_0 c^2 \partial^2 + 2Q^2 \phi_0^* \phi_0) A^\mu = 0 \\ \partial_\nu A^\nu &= 0, \qquad \left(-\partial^2 + \frac{2Q^2 \phi_0^* \phi_0}{\epsilon_0 c^2}\right) A^\mu = 0 \end{aligned} \tag{533}$$

which shows that the components of A_μ satisfy a massive wave equation with mass parameter $\mu^2 = 2Q^2 \phi_0^* \phi_0 / \epsilon_0 c^2 = 2\mu_0 Q^2 |\phi_0|^2$. So the question is: under what circumstances such a constant solution exists? Plugging a constant ansatz into the equation for ϕ at $A = 0$ shows that one must have $\phi_0 U'(|\phi_0|^2) = 0$. That is $|\phi_0| > 0$ must be an extremum of U. A simple example would be $U = \lambda(|\phi|^2 - |\phi_0|^2)^2/4$. If U' were a (positive) constant the only constant solution would be $\phi = 0$. It is true that $\phi U'$ is also zero at $\phi = 0$ in the example, but the value of U at $\phi = 0$ is higher than at $\phi = \phi_0$. Of course a finite difference in energy density translates to a total energy difference proportional to the volume of the system, a very large energy difference indeed!

After finding a constant solution, we can then describe the excitations by writing $\phi = \phi_0 + h + ia$ and then expanding the action in powers of A_ν, h. One can conveniently choose $a = 0$ as a gauge condition. Then

$$\begin{aligned} U &= \lambda(h^2 + 2\phi_0 h)^2/4 = \lambda \phi_0^2 h^2 + O(h^3) \\ \mathcal{L} &= -\frac{\epsilon_0}{4} F_{\rho\sigma} F^{\rho\sigma} - Q^2 \phi_0^2 A^2 - (\partial h)^2 - \lambda \phi_0^2 h^2 + O(A^2 h, h^3) \end{aligned} \tag{534}$$

So the theory describes a massive photon, with mass $M_A = \hbar\mu/c$ or $M_A^2 = 2\hbar^2 Q^2 \phi_0^2 / \epsilon_0 c^4$, interacting with a massive scalar $M_h^2 = \hbar^2 \lambda \phi_0^2 / c^2$. The Higgs mechanism is important in elementary particle physics not to give a mass to the photon, for which there is no evidence, but rather to give mass to the gauge bosons that mediate the weak interactions, which are extremely short range. Indeed the effect of the mass term on the analogue of the Coulomb potential is dramatic. Instead of Poisson's equation one has

$$(-\nabla^2 + \mu^2)\phi = \delta(\boldsymbol{r}), \qquad \phi = \frac{e^{-\mu r}}{4\pi r} \tag{535}$$

The weak interaction vector bosons have a mass roughly 100 times the proton mass, which corresponds to $\mu \sim 1/(10^{-16}\text{cm})$! A particle associated with the scalar field h was discovered in 2012. Finding the "Higgs" particle was one of the goals of the large hadron collider (LHC) still running at CERN in Geneva. From the quartic potential model for U the Higgs $(\text{mass})^2$ is $M_{\rm H}^2 = \hbar^2 \lambda \phi_0^2 / c^2$ compared to the vector boson mass (Note that $Q = e/\hbar$, where $-e$ is the electron charge.) $M_{\rm Z}^2 = 2e^2 \phi_0^2 / \epsilon_0 c^4$. The ratio $M_{\rm H}^2 / M_{\rm Z}^2 = \lambda \epsilon_0 \hbar^2 c^2 / 2e^2 = \lambda \hbar c / 8\pi\alpha$ was not determined by the properties of any of the particles previously observed. But notice that if $M_{\rm H} > 10 M_{\rm Z}$, $\lambda\hbar c/8\pi > 100\alpha = O(1)$, and the electroweak theory would cease to be perturbative. Fortunately for theoretical physics, M_H waas measured to be only 125 GeV$\approx 1.4 M_Z$ so $\lambda \hbar c / 8\pi \approx 2\alpha \ll 1$.

In condensed matter physics, the Higgs mechanism is in essence a model of superconductivity, which is caused by the condensation, at low enough temperature, of pairs of electrons (Cooper pairs) which are quasi bound states with total spin $S = 0$ and total charge $q = Q\hbar = -2e$. In this case the scalar field provides a phenomenological description of the Cooper pair condensate. The energy density difference $U(0) - U(\phi_0^2) = \lambda\phi_0^4/4$ models the energy gain when the pairs condense, which is reflected in the scalar field acquiring a finite background value $\phi_0 \neq 0$. In this application the scalar field is only present in the material. The induced current in the material, in the approximation that the scalar field is just its uniform background value, is just $J_\rho \approx -2Q^2\phi_0^2 A_\rho$. Its conservation in this approximation specifies the Lorenz gauge condition $\partial_\rho A^\rho = 0$.

We can relate this form for the current to the effective magnetic permeability of the superconducting material. Recall that for a static situation, the magnetization $\boldsymbol{M}$ is related to the bound current by $\boldsymbol{J} = \nabla \times \boldsymbol{M}$. Thus for our model superconductor we can write

$$\begin{aligned} \nabla \times (\nabla \times \boldsymbol{M}) &= -2Q^2\phi_0^2 \nabla \times \boldsymbol{A} \\ -\nabla^2 \boldsymbol{M} + \nabla(\nabla \cdot \boldsymbol{M}) &= -2Q^2\phi_0^2 \boldsymbol{B} \end{aligned} \tag{536}$$

Remembering that $\boldsymbol{M} = \boldsymbol{B}/\mu_0 - \boldsymbol{H} = (\mu_0^{-1} - \mu^{-1})\boldsymbol{B}$. This means that in Fourier space, assuming for simplicity that $\nabla \cdot \boldsymbol{M} = 0$, $(\mu_0^{-1} - \mu^{-1}) = -2Q^2\phi_0^2/\boldsymbol{k}^2$, or $\mu(\boldsymbol{k}^2) = \mu_0 \boldsymbol{k}^2/(\boldsymbol{k}^2 + 2\mu_0 Q^2\phi_0^2)$. In particular $\mu \to 0$ for $\boldsymbol{k} = 0$, i.e. for a uniform applied field. This is why we say that a superconductor is a perfect diamagnet. The fact that $\mu \neq 0$ for finite $\boldsymbol{k}$ means that a spatially varying magnetic field would be allowed to exist to some extent within the superconductor. Notice that $\mu \to \mu_0$ at very short wavelength ($\boldsymbol{k} \to \infty$).

To understand this application a little better, let's consider the interface between a "superconductor" filling the half space $z < 0$, with a static uniform electric or magnetic field in otherwise empty space for $z > 0$. The static vector potential satisfies

$$(-\nabla^2 + M^2)A^\mu = 0, \quad z < 0; \qquad -\nabla^2 A^\mu = 0, \quad z > 0 \tag{537}$$

At the interface $z = 0$ we require that A^μ and $\partial A^\mu/\partial z$ be continuous. A solution with $A^\mu(z)$ a function of z only is easily found:

$$A^\mu = a^\mu e^{Mz}, \quad z < 0; \qquad A^\mu(z) = a^\mu(Mz + 1), \quad z > 0 \tag{538}$$

the Lorenz gauge condition requires $\boldsymbol{a} \cdot \hat{z} = 0$. With this solution the uniform fields for $z > 0$ are $\boldsymbol{E} = -cMa^0\hat{z}$ and $\boldsymbol{B} = \hat{z} \times \boldsymbol{a}$. Thus they automatically fulfill the conditions $\boldsymbol{E}_t = 0, \boldsymbol{B}_n = 0$ expected of a perfect conducting diamagnet. However, with M finite, the fields penetrate into the superconductor, a distance M^{-1} called the London penetration depth. The mass of the scalar determines another length scale characterizing the "size" of the scalar particle. In conventional superconductivity this would be the size of the Cooper pair which is actually quite large compared to the atomic size. This second length scale is called the coherence length.

Meissner Effect

The expulsion of magnetic field from the interior of a superconductor is the Meissner effect. At a rough level postulating that $\mu = 0$ in superconducting material accounts for the effect. We have just seen that interpreting the effect in terms of an effective photon mass incorporates the more subtle effect of a finite penetration depth for the magnetic field. But so far we have not allowed the scalar field to vary spatially, though the field equations certainly allow it. Superconductivity occurs when $\phi = \phi_0$, but we can imagine a region in which superconductivity is destroyed because $\phi \approx 0$ in that region. The energy cost of destroying superconductivity in a volume V is $\Delta E = [U(0) - U(|\phi_0|^2)]V \to \lambda\phi_0^4 V/4$ for our quartic model. This energy difference determines a critical magnetic field H_c via

$$\frac{1}{2}\mu_0 H_c^2 \equiv U(0) - U(|\phi_0|^2). \tag{539}$$

For $H > H_c$ in a region of superconductor, there is enough magnetic energy to drive that region of the superconductor normal. Of course, at the boundary of this region, the field ϕ should smoothly vary from 0 to ϕ_0, so there will be some surface energy arising from the $(\nabla\phi)^2$ term in the energy. To go further we need to look to the field equations.

Magnetic Flux tubes

Let us consider static solutions ($\dot{\phi} = \dot{\boldsymbol{A}} = 0$) of the equations of motion, with $\boldsymbol{E} = 0$ ($A_0 = 0$). In this case the energy density of the system is proportional to the Lagrangian density, so Hamilton's principle for static solutions is equivalent to extremizing the total energy, the density of which reduces to

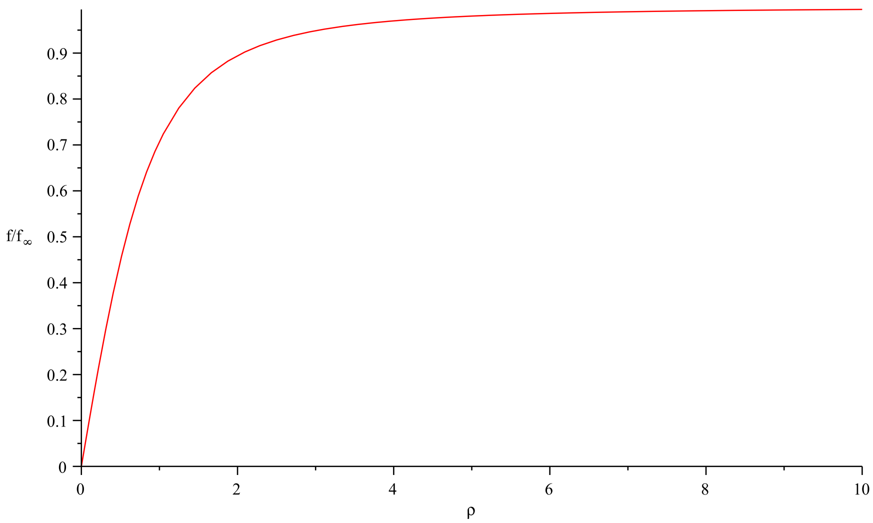
$$T^{00} \to \frac{\epsilon_0}{2}c^2\boldsymbol{B}^2 + (\nabla + iQ\boldsymbol{A})\phi^* \cdot (\nabla - iQ\boldsymbol{A})\phi + \frac{\lambda}{4}(\phi^*\phi - \phi_0^2)^2 \tag{540}$$

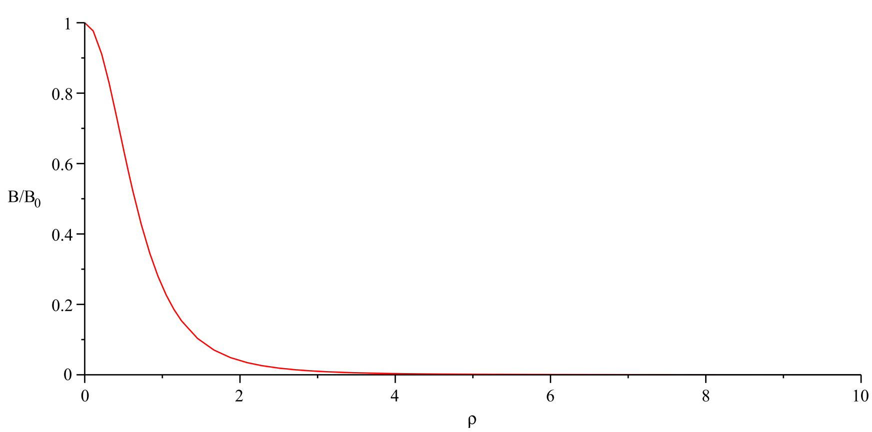
where we have assumed the simple quartic model for U. A very interesting solution describes a magnetic flux vortex tube which penetrates the superconductor. This solution describes a real physical phenomenon in which magnetic fields penetrate a finite sized superconductor through narrow tubes in which the superconductor has been driven normal. To make a tractable problem we assume that the superconductor fills all of space and that the vortex tube is centered on the z-axis and extends from $z = -\infty$ to $z = +\infty$. All the fields will be assumed independent of z, and we will be interested in the energy per unit length. Since we are looking for a static z-independent solution it is enough to extremize the energy per unit length for the axially symmetric situation you worked out in Problem Set 4:

$$T = 2\pi\int_0^\infty d\rho\left[\epsilon_0 c^2\frac{[(\rho A)']^2}{2\rho} + \rho f'^2 + \rho\left(\frac{m}{\rho} - QA\right)^2 f^2 + \rho\frac{\lambda}{4}(f^2 - \phi_0^2)^2\right] \tag{541}$$

where $\boldsymbol{A} = A(\rho)\hat{\phi}$ and $\phi = f(\rho)e^{im\varphi}$. As you showed, such a potential produces a magnetic field $\boldsymbol{B} = [\rho A]'/\rho$.

In Set 5 I have asked you to find an estimate for T based on the variational principle. Here we will discuss the necessary conditions on the fields at $\rho \to 0, \infty$. Since each of the four terms in the integrand is positive, there are 4 independent integrals that must be finite. At $\rho \to \infty$ we must have $f \to \phi_0$ and $QA \sim m/\rho$. By Stokes theorem, the last behavior shows that the magnetic flux $\Phi_B = \oint d\boldsymbol{l} \cdot \boldsymbol{A} = 2\pi m/Q$. This is why we call this solution a magnetic flux tube. In order for the first integral to converge at $\rho = 0$ you will show that necessarily $A \to 0$ as $\rho \to 0$. If this is so the convergence of the third term requires $f(0) = 0$. Thus on the axis of the flux tube the superconductor has been driven normal. If all these conditions are met T will be finite. It then remains to find its minimum for each integer value m. Clearly if $m = 0$ the minimum is for $f = \phi_0, A = 0$ for which $T = 0$. This is just the ground state of the superconductor. For $m \neq 0$ the minimum is for a flux tube with total flux $2\pi m/Q$. Without going any further we know a lot about the solution. As ρ varies from 0 to ∞ f rises from 0 to ϕ_0. Meanwhile $B(\rho)$ falls from some nonzero value to zero. Your exercise is to flesh out these qualitative conclusions with a concrete variational approximate solution.





9 Propagation of Plane waves in Materials

9.1 Oscillator model for frequency dependence of a dielectric

The dielectric constant and magnetic permeability describe the response of a material to applied fields. As such they depend on two space-time points: the location and time the field is applied and the location and time the response is measured. For example, in full generality we should write, in the linear response approximation,

$$E^k(x) = \int d^4y \epsilon^{-1}_{kj}(x,y) D^j(y), \qquad B^k(x) = \int d^4y \mu_{kj}(x,y) H^j(y) \tag{542}$$

We shall generally be interested in homogeneous and isotropic materials, in which case $\epsilon^{-1}_{kj} = \delta_{kj}\epsilon$ and $\mu_{kj} = \delta_{kj}\mu$ will depend only on $x - y$, but the relationship will still, in general, be non-local in spacetime

$$\boldsymbol{E}(x) = \int d^4y \epsilon^{-1}(x-y) \boldsymbol{D}(y), \qquad \boldsymbol{B}(x) = \int d^4y \mu(x-y) \boldsymbol{H}(y) \tag{543}$$

Convolution integrals of this type can be diagonalized by simple Fourier transforms:

$$\begin{aligned} \boldsymbol{E}(x) &= \int \frac{d^4k}{(2\pi)^4} \tilde{\boldsymbol{E}}(k) e^{ik\cdot x}, \qquad \boldsymbol{D}(x) = \int \frac{d^4k}{(2\pi)^4} \tilde{\boldsymbol{D}}(k) e^{ik\cdot x} \\ \epsilon^{-1}(x-y) &= \int \frac{d^4k}{(2\pi)^4} \tilde{\epsilon}^{-1}(k) e^{ik\cdot(x-y)} \end{aligned} \tag{544}$$

and similarly for $\boldsymbol{B}, \boldsymbol{H}, \mu$. Then we have

$$\tilde{\boldsymbol{E}}(k) = \tilde{\epsilon}^{-1}(k) \tilde{\boldsymbol{D}}(k), \qquad \tilde{\boldsymbol{B}}(k) = \tilde{\mu}(k) \tilde{\boldsymbol{H}}(k) \tag{545}$$

We shall not attempt a serious treatment of the theory of matter in this course. Instead we will use simplified models. In an insulator, all of the charged particles are locally bound within atoms and molecules. The main qualitative fact we need to account for is that if an electron is pulled by the action of an electric field, say, the binding mechanism supplies a restoring force. For many purposes an adequate model for the restoring force is a damped harmonic oscillator force. Then the equation of motion for the electron in an applied electric field, assuming nonrelativistic motion, would be

$$\ddot{\boldsymbol{r}} + \gamma \dot{\boldsymbol{r}} + \omega_0^2 \boldsymbol{r} = -\frac{e}{m} \boldsymbol{E}(\boldsymbol{r}, t) \tag{546}$$

We make the simplifying assumption that $\boldsymbol{E} \approx \boldsymbol{E}_0 e^{-i\omega t}$, so that the steady state solution is $\boldsymbol{r} = (-e/m)(\omega_0^2 - \omega^2 - i\gamma\omega)^{-1}\boldsymbol{E}$. If there are N molecules per unit volume and each molecule has Z electrons f_j of which have binding frequency ω_j and damping parameter γ_j, so $\sum_j f_j = Z$, then the dipole moment density will be $\boldsymbol{P} = (e^2/m)N\sum_j f_j(\omega_j^2 - \omega^2 - i\gamma_j\omega)^{-1}\boldsymbol{E}$. Then $\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} \equiv \epsilon \boldsymbol{E}$ implies that

$$\epsilon(\omega) = \epsilon_0 + \frac{e^2N}{m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega} = \epsilon_0 + \frac{e^2N}{m} \sum_j \frac{f_j(\omega_j^2 - \omega^2 + i\gamma_j\omega)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2} \tag{547}$$

The essential structure of the frequency dependence given by this simple minded model is a feature of more serious quantum mechanical calculations, where the ω_j, f_j, γ_j are effective quantum mechanical parameters. The resonance behavior given by the denominators is an ubiquitous feature of metastable excited states in quantum mechanics.

The consequences of such frequency dependence for a plane wave $e^{ikz-i\omega t}$ propagating in the dielectric can be read off from the relation $k = \omega\sqrt{\epsilon\mu} \to (\omega/c)\sqrt{\epsilon(\omega)/\epsilon_0}$. Since ϵ is complex, the wave number will also be complex $k = \beta + i\alpha/2$:

$$\beta^2 - \frac{\alpha^2}{4} = \frac{\omega^2}{c^2}\text{Re}\,\frac{\epsilon}{\epsilon_0}, \qquad \alpha\beta = \frac{\omega^2}{c^2}\text{Im}\,\frac{\epsilon}{\epsilon_0} \tag{548}$$

$$\text{Re}\,\epsilon = \epsilon_0 + \frac{e^2N}{m}\sum_j \frac{f_j(\omega_j^2-\omega^2)}{(\omega_j^2-\omega^2)^2+\gamma_j^2\omega^2} \tag{549}$$

$$\text{Im}\,\epsilon = \frac{e^2N}{m}\sum_j \frac{f_j\gamma_j\omega}{(\omega_j^2-\omega^2)^2+\gamma_j^2\omega^2} \tag{550}$$

Thus the imaginary part of ϵ, which is manifestly positive, implies an attenuation factor $e^{-\alpha z/2}$ multiplying the phase of the plane wave. This represents a loss of energy from the plane wave to the medium. Since the model result for Im ϵ is proportional to the damping coefficients of the oscillator, this outcome makes good sense.

For $\omega < \omega_j$ for all j, Re ϵ is manifestly larger than ϵ_0, corresponding to a n index of refraction larger than 1. Assuming none of the ω_j vanish, limit $\omega \to 0$ reads

$$\epsilon(0) = \epsilon_0 + \frac{e^2N}{m}\sum_j \frac{f_j}{\omega_j^2} \tag{551}$$

which is characteristic of an insulator. However as ω increases past the resonant frequencies, more and more terms become negative, until eventually ϵ becomes less than ϵ_0. Far above the largest ω_j, $\epsilon \sim \epsilon_0(1-\omega_p^2/\omega^2)$ where the plasma frequency is given by $\omega_p^2 = N_e e^2/\epsilon_0 m$, where $N_e = NZ$ is just the total number density of electrons in the material. For such high frequencies, the restoring forces have no time to act and all electrons are effectively free. For insulating materials, this characteristic form only applies for $\omega \gg \omega_p$, so that the index of refraction is only slightly less than 1.

9.2 Conductivity

The simple resonance model discussed above can be used as a model of conductivity if one of the ω_j, say $\omega_0 = 0$. Then separating their contribution from the rest gives

$$\begin{aligned}\epsilon(\omega) &= \epsilon_0 + \frac{e^2N}{m}\sum_j \frac{f_j}{\omega_j^2-\omega^2-i\gamma_j\omega} - \frac{e^2N}{m\omega}\frac{f_0}{\omega+i\gamma_0} \equiv \epsilon_b(\omega) - \frac{e^2Nf_0}{m\omega(\omega+i\gamma_0)}\\ &\sim i\frac{e^2N_e^{free}}{m\omega\gamma_0}, \qquad \text{as } \omega \to 0\end{aligned} \tag{552}$$

where we have identified $Nf_0 \equiv N_e^{free}$ as the number density of free (i.e. unbound) electrons. To interpret this behavior, notice that the displacement current

$$\partial \boldsymbol{D}/\partial t \to -i\omega\tilde{\boldsymbol{D}} = -i\omega\epsilon(\omega)\tilde{\boldsymbol{E}} = -i\omega\epsilon_b(\omega)\tilde{\boldsymbol{E}} + \frac{e^2 N_e^{free}}{m(\gamma_0 - i\omega)}\tilde{\boldsymbol{E}} \tag{553}$$

The last term can be interpreted in the zero frequency limit as Ohm's law

$$\frac{e^2 N_e^{free}}{m(\gamma_0 - i\omega)}\tilde{\boldsymbol{E}} \to \sigma\tilde{\boldsymbol{E}}, \qquad \sigma = \frac{e^2 N_e^{free}}{m\gamma_0} \tag{554}$$

So this model reproduces Ohm's law in the quasi-static limit. In the formula for the conductivity, $1/\gamma_0$ can be interpreted as the mean time τ that an electron travels freely between collisions with impurities or lattice perturbations. Then the average drift velocity in an electric field would be $\boldsymbol{v}_d = -e\boldsymbol{E}\tau/m$, so $\boldsymbol{J} = -eN_e^{free}\boldsymbol{v}_d = (e^2 N_e^{free}\tau/m)\boldsymbol{E} \equiv \sigma\boldsymbol{E}$.

We can write the dielectric constant for a conductor in terms of the effective plasma frequency $\omega_p^2 \equiv e^2 N_e^{free}/m\epsilon_0$,

$$\omega_p^2 = \frac{e^2 N_e^{free}}{m\epsilon_0} \tag{555}$$

$$\epsilon(\omega) = \epsilon_b(\omega) - \epsilon_0\frac{\omega_p^2}{\omega(\omega + i\gamma_0)} \tag{556}$$

Since ω_p is typically many orders of magnitude larger than γ_0, there is a range of frequencies for which the second term is real negative and large. In that range the dielectric constant is negative and the wave number $k = \omega\sqrt{\epsilon/\epsilon_0}/c$ is imaginary meaning that em waves cannot enter the conductor. In this frequency range light incident on such a metal is completely reflected. However at some critical frequency $\epsilon(\omega_c) = 0$ and above that value the metal starts to transmit radiation reducing the reflectivity drastically.

9.3 Plasmas and the Ionosphere

An ionized gas is called a plasma. For a dilute (tenuous) plasma the charged particles are essentially free with negligible damping. In our resonance model none of the $\omega_j \neq 0$ appear and the formula collapses to

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma_0)} \approx 1 - \frac{\omega_p^2}{\omega^2} \tag{557}$$

where the approximate form neglects damping. This is of course the high frequency behavior for all materials, but for a tenuous plasma it is also valid for $\omega << \omega_p$ for which $\epsilon < 0$. Then

$$kc = \omega\sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{\omega^2 - \omega_p^2} \to i\omega_p, \qquad \alpha = 2\frac{\omega_p}{c} \tag{558}$$

In typical laboratory plasmas α^{-1} is .002 cm to .2 cm.

If we rewrite the relation between frequency and wave number in the form $\omega^2 = \omega_p^2 + k^2c^2$, we see that the group velocity of a wave packet propagating in a plasma will be $\boldsymbol{v}_g = \nabla_k\omega(\boldsymbol{k}) = c^2\boldsymbol{k}/\omega(\boldsymbol{k})$. Traveling waves with frequency $\omega > \omega_p$ are supported by a plasma, and they travel at a speed less than light. Oscillations at the plasma frequency can exist when $\boldsymbol{k} = 0$, i.e. these disturbances are at rest in the plasma.

Plasma in a Magnetic field In the ionosphere the presence of the earth's magnetic field alters the propagation of waves in an interesting way. To see how we return to the equation of motion for a particle in a background uniform magnetic field in addition to the em wave:

$$\begin{aligned} m\ddot{\boldsymbol{r}} &= -e\dot{\boldsymbol{r}} \times \boldsymbol{B}_0 - e\boldsymbol{E}e^{-i\omega t} \\ -m\omega^2\boldsymbol{r}_0 &= ie\omega\boldsymbol{r}_0 \times \boldsymbol{B}_0 - e\boldsymbol{E} \end{aligned} \tag{559}$$

To solve this equation first note that dotting both sides with $\boldsymbol{B}_0$ yields $\boldsymbol{r}_0\cdot\boldsymbol{B}_0 = e\boldsymbol{E}\cdot\boldsymbol{B}_0/m\omega^2$. Then taking the cross product of both sides with $\boldsymbol{B}_0$ gives;

$$\begin{aligned} -m\omega^2\boldsymbol{r}_0 \times \boldsymbol{B}_0 &= ie\omega(\boldsymbol{B}_0\boldsymbol{r}_0 \cdot \boldsymbol{B}_0 - \boldsymbol{B}_0^2\boldsymbol{r}_0) - e\boldsymbol{E} \times \boldsymbol{B}_0 \\ &= ie\omega(\boldsymbol{B}_0 e\boldsymbol{E} \cdot \boldsymbol{B}_0/m\omega^2 - \boldsymbol{B}_0^2\boldsymbol{r}_0) - e\boldsymbol{E} \times \boldsymbol{B}_0 \end{aligned}$$

Plugging this back into the first equation then yields

$$\begin{aligned} \frac{-m\omega^2}{ie\omega}(e\boldsymbol{E} - m\omega^2\boldsymbol{r}_0) &= ie\omega(\boldsymbol{B}_0 e\boldsymbol{E} \cdot \boldsymbol{B}_0/m\omega^2 - \boldsymbol{B}_0^2\boldsymbol{r}_0) - e\boldsymbol{E} \times \boldsymbol{B}_0 \\ \frac{m\omega^2}{ie\omega}(m\omega^2\boldsymbol{r}_0) + ie\omega(\boldsymbol{B}_0^2\boldsymbol{r}_0) &= \frac{m\omega^2}{ie\omega}e\boldsymbol{E} + \frac{ie^2\omega}{m\omega^2}\boldsymbol{B}_0(\boldsymbol{E} \cdot \boldsymbol{B}_0) - e\boldsymbol{E} \times \boldsymbol{B}_0 \end{aligned}$$

Introduce $\boldsymbol{\omega}_B = e\boldsymbol{B}_0/m$ and find

$$\begin{aligned} (\omega^2 - \omega_B^2)\boldsymbol{r}_0 &= \frac{e}{m}\boldsymbol{E} - \frac{e}{m\omega^2}(\boldsymbol{\omega}_B\boldsymbol{E} \cdot \boldsymbol{\omega}_B) - \frac{ie}{m\omega}\boldsymbol{E} \times \boldsymbol{\omega}_B \\ \boldsymbol{P} &= -eN\boldsymbol{r}_0 = -\epsilon_0\frac{\omega_p^2}{\omega^2 - \omega_B^2}\left(\boldsymbol{E} - i\boldsymbol{E} \times \frac{\boldsymbol{\omega}_B}{\omega} - \frac{\boldsymbol{\omega}_B}{\omega}\frac{\boldsymbol{\omega}_B}{\omega} \cdot \boldsymbol{E}\right) \qquad (560) \\ \boldsymbol{D} &= \epsilon_0\boldsymbol{E} + \boldsymbol{P} \\ &= \epsilon_0\left[\boldsymbol{E} - \frac{\omega_p^2}{\omega^2 - \omega_B^2}\left(\boldsymbol{E} - i\boldsymbol{E} \times \frac{\boldsymbol{\omega}_B}{\omega} - \frac{\boldsymbol{\omega}_B}{\omega}\frac{\boldsymbol{\omega}_B}{\omega} \cdot \boldsymbol{E}\right)\right] \qquad (561) \end{aligned}$$

This is an example where the effective dielectric constant is a matrix, not just a number.

We next use Maxwell's equations to determine how a plane wave propagates through this medium. Plugging Faraday's law for a plane wave $\boldsymbol{B} = \boldsymbol{k} \times \boldsymbol{E}/\omega$ into the Ampere-Maxwell equation yields

$$\begin{aligned} \boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}) &= -\omega^2\mu_0\boldsymbol{D} \\ k^2(\boldsymbol{E} - \hat{k}\hat{k} \cdot \boldsymbol{E}) &= \frac{\omega^2}{c^2}\left[\boldsymbol{E} - \frac{\omega_p^2}{\omega^2 - \omega_B^2}\left(\boldsymbol{E} - i\boldsymbol{E} \times \frac{\boldsymbol{\omega}_B}{\omega} - \frac{\boldsymbol{\omega}_B}{\omega}\frac{\boldsymbol{\omega}_B}{\omega} \cdot \boldsymbol{E}\right)\right] \end{aligned} \tag{562}$$

The right side of this equation dotted into $\hat{k}$ must be zero, which puts a constraint on components of the electric field:

$$
\begin{aligned}
0 &= \hat{k}\cdot\left[\boldsymbol{E}-\frac{\omega_p^2}{\omega^2-\omega_B^2}\left(\boldsymbol{E}-i\boldsymbol{E}\times\frac{\boldsymbol{\omega}_B}{\omega}-\frac{\boldsymbol{\omega}_B}{\omega}\frac{\boldsymbol{\omega}_B}{\omega}\cdot\boldsymbol{E}\right)\right]\\
&= \boldsymbol{E}\cdot\left[\hat{k}-\frac{\omega_p^2}{\omega^2-\omega_B^2}\left(\hat{k}-i\frac{\boldsymbol{\omega}_B}{\omega}\times\hat{k}-\frac{\hat{k}\cdot\boldsymbol{\omega}_B}{\omega}\frac{\boldsymbol{\omega}_B}{\omega}\right)\right] \qquad (563)
\end{aligned}
$$

(When $B=0$ this is simply the constraint that $\boldsymbol{E}$ is perpendicular to the propagation direction. But when $B\neq 0$ it causes complications in the solution for general $\boldsymbol{k},\boldsymbol{\omega}_B$.

We consider two simpler cases. First consider propagation parallel to the magnetic field, $\hat{k}=\boldsymbol{\omega}_B/\omega_B$. Then it easily follows (assuming $\omega\neq\omega_p$) that the constraint becomes $\boldsymbol{k}\cdot\boldsymbol{E}=0$, and the equation reduces to

$$
\boldsymbol{k}^2\boldsymbol{E} = \frac{\omega^2}{c^2}\left[\boldsymbol{E}-\frac{\omega_p^2}{\omega^2-\omega_B^2}\left(\boldsymbol{E}-i\frac{\omega_B}{\omega}\boldsymbol{E}\times\hat{k}\right)\right] \qquad (564)
$$

We want to find two (necessarily complex) solutions for $\boldsymbol{E}$ such that $\boldsymbol{E}\times\hat{k}$ is a multiple of $\boldsymbol{E}$. Taking $\boldsymbol{k}$ in the z-direction, one notes that $(\hat{x}\pm i\hat{y})\times\hat{z}=\pm i(\hat{x}\pm i\hat{y})$ so that the eigenvectors are the circularly polarized states and the relation between k and ω is different for each:

$$
k_\mp^2=\frac{\omega^2}{c^2}\left[1-\frac{\omega_p^2}{\omega^2-\omega_B^2}\left(1\pm\frac{\omega_B}{\omega}\right)\right]\equiv\frac{\omega^2}{c^2}\frac{\epsilon_\mp}{\epsilon_0} \qquad (565)
$$

$$
\boldsymbol{E}_\mp = \frac{\hat{x}\pm i\hat{y}}{\sqrt{2}}, \qquad \epsilon_\mp=\epsilon_0\left(1-\frac{\omega_p^2}{\omega(\omega\mp\omega_B)}\right) \qquad (566)
$$

Notice that the relative phase of the two polarization states will not be maintained in space. In particular, if $\boldsymbol{E}$ starts out at $z=0$ linearly polarized, say in the $\boldsymbol{x}$ direction, then at $z\neq 0$, we have

$$
\boldsymbol{E}=\frac{E}{2}[(\hat{x}+i\hat{y})e^{i\omega n_- z/c}+(\hat{x}-i\hat{y})e^{i\omega n_+ z/c}]=Ee^{i\omega(n_++n_-)z/2c}[\hat{x}\cos\theta(z)+\hat{y}\sin\theta(z)] \quad (567)
$$

where $\theta(z)=\omega z(n_+-n_-)/2c$. Thus the direction of linear polarization rotates with z (Faraday rotation).

The second simplifying assumption is to take a general propagation direction but assume $\omega_B\ll\omega$. Then the effect of the magnetic field is small, and it is appropriate to perturb around $B=0$. With $\boldsymbol{k}=k\hat{z}$, the two zeroth order solutions are $\boldsymbol{E}_{0\mp}=(\hat{x}\pm i\hat{y})$ with degenerate $k_0^2=(\omega^2-\omega_p^2)/c^2$. Then the first order equation reads

$$
(k^2-k_0^2)\boldsymbol{E}_{0\mp}+k_0^2(\boldsymbol{E}_{1\mp}-\hat{k}\hat{k}\cdot\boldsymbol{E}_{1\mp}) = k_0^2\boldsymbol{E}_{1\mp}+\frac{\omega^2}{c^2}\left[-\frac{\omega_p^2}{\omega^2}\left(-i\boldsymbol{E}_{0\mp}\times\frac{\boldsymbol{\omega}_B}{\omega}\right)\right]
$$

Taking the scalar product of both sides with $\hat{x} \mp i\hat{y}$, the $\boldsymbol{E}_1$ terms drop out, and one finds

$$\begin{aligned} 2(k^2 - k_0^2) &= \frac{\omega_p^2}{c^2}(\hat{x} \mp i\hat{y}) \cdot \left(i(\hat{x} \pm i\hat{y}) \times \frac{\boldsymbol{\omega}_B}{\omega} \right) = i\frac{\omega_p^2}{c^2}[(\hat{x} \mp i\hat{y}) \times (\hat{x} \pm i\hat{y})] \cdot \frac{\boldsymbol{\omega}_B}{\omega} \\ k^2 &= k_0^2 \mp \frac{\omega_p^2}{c^2}\frac{\hat{z} \cdot \boldsymbol{\omega}_B}{\omega} = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2} \mp \frac{\omega_p^2}{\omega^2}\frac{\hat{z} \cdot \boldsymbol{\omega}_B}{\omega}\right) \qquad (568) \\ \epsilon_{\mp} &= 1 - \frac{\omega_p^2}{\omega^2}\left(1 \pm \frac{\hat{z} \cdot \boldsymbol{\omega}_B}{\omega}\right) \qquad (569) \end{aligned}$$

which agrees with the first case when the magnetic field is in the z-direction.

9.4 Group Velocity

The frequency dependence of ϵ leads to a frequency dependent group velocity as we have already discussed in the context of wave packets. In our previous discussion our packets were treated as superpositions of a narrow band of wave numbers with the frequency a function of $\boldsymbol{k}$. Then the formula for the group velocity is $\boldsymbol{v}_g = \nabla_k \omega(\boldsymbol{k})$. In our discussion of frequency dependence we have presented the wave number as a function of frequency $kc = \omega n(\omega)$ where $n = \sqrt{\epsilon\mu/\epsilon_0\mu_0}$ is the index of refraction. Then we can express the group velocity in terms of frequency derivatives: $c = v_g(n + \omega dn/d\omega)$ or

$$v_g = \frac{c}{n + \omega dn/d\omega} \qquad (570)$$

An interesting example is the ϵ_- mode of a plasma with magnetic background field. We have in the low frequency limit

$$n_- \approx \frac{\omega_p}{\sqrt{\omega\omega_B}}, \qquad n_- + \omega\frac{dn_-}{d\omega} \approx \frac{\omega_p}{2\sqrt{\omega\omega_B}}, \qquad v_g = c\frac{2\sqrt{\omega\omega_B}}{\omega_p} \qquad (571)$$

Lower frequency components travel more slowly, so a broad frequency pulse will be received smeared out in time with the later arrivals lower in frequency (whistlers).

In our resonance model of dispersion (i.e. frequency dependent index of refraction), ϵ and hence n increase with frequency except for narrow windows surrounding the resonance locations ω_j. This is called normal dispersion and from the formula for the group velocity we see that $v_g < c/n$ for normal dispersion. However, as one passes through a resonance, the index of refraction starts falling, and for narrow resonances there is a narrow range where the decline is rapid, $dn/d\omega$ is large and negative (anomalous dispersion). In a region of anomalous dispersion v_g can get large, even larger than the speed of light–it can even become negative! There is no violation of causality though. The interpretation of v_g as the speed of a signal requires that the frequency width of the packet is much smaller than the scale over which the integrand varies. But in regions of anomalous dispersion the integrand is rapidly varying so the width has to be extremely tiny corresponding to a huge spread in the duration of the packet in time and a corresponding fuzziness in the location of any wave front. To establish an actual signal velocity there must be a sharp wave front, which requires a broad range of frequencies. The regions of anomalous dispersion contribute very little to such a broad packet.

The variation in the group velocity with frequency produces a spreading of the wave packet. The current homework assignment includes a problem where this can be explicitly seen using a Gaussian wave packet. But the phenomenon is inevitable because a band of wave numbers Δk has a spread of group velocities $\Delta v_g = \Delta k d^2\omega/dk^2$, and an additional contribution to the width in space of $\Delta k t d^2\omega/dk^2$. The initial width is at best $1/\Delta k$, so the width after a time t is at least

$$\Delta x(t) = \sqrt{\frac{1}{\Delta k^2} + \Delta k^2 t^2 \left(\frac{d^2\omega}{dk^2}\right)^2} \tag{572}$$

This matches the behavior you found for a Gaussian wave packet. Notice that the narrower the band of wave numbers, the longer it takes for the spreading to take effect. Of course at the same time the initial width of the packet is correspondingly large. In scattering experiments the only practical limit on the size of the wave-packets is the size of the apparatus.

9.5 Causality and Dispersion Relations

As we have mentioned the relation between $\boldsymbol{D}$ and $\boldsymbol{E}$ in the linear response approximation is a nonlocal one of the form

$$\boldsymbol{D}(\boldsymbol{r},t) = \epsilon_0 \boldsymbol{E}(\boldsymbol{r},t) + \epsilon_0 \int d^3r' dt' G(\boldsymbol{r}-\boldsymbol{r}', t-t')\boldsymbol{E}(\boldsymbol{r}',t') \tag{573}$$

where we have further assumed translation invariance in space and time. Here G is the Fourier transform of the wave number and frequency dependent dielectric constant

$$\epsilon_0 G(\boldsymbol{r},t) = \int \frac{d^3q d\omega}{(2\pi)^4} (\epsilon(\boldsymbol{q},\omega) - \epsilon_0) e^{i\boldsymbol{q}\cdot\boldsymbol{r} - i\omega t} \tag{574}$$

The physical interpretation is that a field applied at $\boldsymbol{r}', t'$ induces a response at $\boldsymbol{r}, t$. In order for causality to be respected, the response cannot occur unless enough time has elapsed for a light signal to travel from $\boldsymbol{r}'$ to $\boldsymbol{r}$. That is

$$G(\boldsymbol{r}-\boldsymbol{r}', t-t') = 0, \qquad \text{unless } c(t-t') > |\boldsymbol{r}-\boldsymbol{r}'| \tag{575}$$

Considering the formula for ϵ

$$\frac{\epsilon(\boldsymbol{q},\omega)}{\epsilon_0} - 1 = \int d^3r \int_{|\boldsymbol{r}|/c}^{\infty} dt G(\boldsymbol{r},t) e^{-i\boldsymbol{q}\cdot\boldsymbol{r} + i\omega t} \tag{576}$$

we see that the dependence on ω can be extended into the upper half complex plane because the $i\omega t$ term in the exponent acquires a negative real part, exponentially improving the convergence of the t integral. Viewed as an analytic function of ω, there will be no singularities in the upper half plane. Singularities in the lower half plane are allowed by causality, because then $i\omega t$ has a positive real part causing worse convergence of the t integration.

In the remaining discussion we ignore the $\boldsymbol{q}$ dependence. Assuming ϵ is independent of $\boldsymbol{q}$ implies that the response is spatially local, i.e.

$$\epsilon_0 G(\boldsymbol{r}, t) = \int \frac{d\omega}{(2\pi)} \, (\epsilon(\omega) - \epsilon_0) e^{-i\omega t} \delta(\boldsymbol{r}) \equiv \epsilon_0 G(t) \delta(\boldsymbol{r}) \tag{577}$$

This will be valid if the distortion of the system due to the applied field is over a region small compared to the scale over which the field varies appreciably. In the case of a wave packet this scale is the typical wavelength.

Let us now recall our resonance model for the frequency dependence of ϵ. Each resonance contribution had the form

$$\frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega} \tag{578}$$

which has pole locations

$$\omega \;=\; -i\frac{\gamma_j}{2} \pm \sqrt{\omega_j^2 - \frac{\gamma_j^2}{4}} \tag{579}$$

and we see that these poles are in the lower half plane for positive γ_j. So in this model causality demands $\gamma_j > 0$. Of course in our model positive γ_j corresponds to a damped as opposed to a runaway oscillator. But it is nice to know that this sort of unphysical runaway behavior is forbidden in models that simply respect causality.

Analyticity is a very powerful mathematical fact about functions. This is manifested by Cauchy's theorem,

$$f(z) = \oint_C \frac{dz'}{2\pi i} \frac{f(z')}{z' - z} \tag{580}$$

where C is a closed curve within a simply connected region of analyticity for $f(z)$.
Mathematical Detour; We can understand Cauchy's theorem in terms of known features of two dimensional elctrostatics. Write $z = x + iy$ and $f = f_1 + if_2$. Then

$$dz f(z) = dx f_1 - dy f_2 + i(dx f_2 + dy f_1) \equiv d\boldsymbol{l} \cdot \boldsymbol{f} + i(d\boldsymbol{l} \cdot \tilde{\boldsymbol{f}}) \tag{581}$$

where the two-vector $\boldsymbol{f} = (f_1, -f_2)^T$ and $\tilde{\boldsymbol{f}} = (f_2, f_1)^T$. Then by Stokes theorem

$$\oint dz f(z) \;=\; \int dA \left([\nabla \times \boldsymbol{f}]_z + i[\nabla \times \tilde{\boldsymbol{f}}]_z \right) \tag{582}$$

But if $f(z) = f(x + iy)$ is analytic it satisfies

$$\begin{aligned} i\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial y} \\ i\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial x} &= \frac{\partial f_1}{\partial y} + i\frac{\partial f_2}{\partial y} \end{aligned} \tag{583}$$

which are the Cauchy-Rieman equations:

$$
\begin{aligned}
\frac{\partial f_1}{\partial x} &= \frac{\partial f_2}{\partial y} \\
-\frac{\partial f_2}{\partial x} &= \frac{\partial f_1}{\partial y}
\end{aligned} \tag{584}
$$

They easily imply that f_1 and f_2 each satisfy the 2D Laplace equation. They also imply

$$
\begin{aligned}
\nabla \times \boldsymbol{f} &= \partial_x(-f_2) - \partial_y f_1 = 0 \\
\nabla \times \tilde{\boldsymbol{f}} &= \partial_x(f_1) - \partial_y f_2 = 0
\end{aligned} \tag{585}
$$

which together give the Cauchy theorem

$$
\oint_C dz f(z) = 9 \tag{586}
$$

for any closed curve C within a simply connected domaain of analyticity. The Cauchy formula follows by first using the Cauchy theorem to deform any closed curve that encloses the point $z' = z$ to an infinitesimal circle C_δ of radius δ centered on the point $z' = z$. Then

$$
\begin{aligned}
\oint_C \frac{dz'}{2\pi i} \frac{f(z')}{z' - z} &= \oint_{C_\delta} \frac{dz'}{2\pi i} \frac{f(z')}{z' - z} \\
&\to \int_0^{2\pi} \frac{id\theta \delta e^{i\theta} f(z)}{2\pi^i \delta e^{i\theta}} = f(z)
\end{aligned} \tag{587}
$$

Dispersion Relations for Dielectrics Because $\epsilon(\omega)$ is analytic in the whole upper half complex ω plane, we can pick C to follow the real axis and close on a large semicircle at $\omega = \infty$. The behavior of $\epsilon(\omega)$ at large ω depends on the behavior of $G(t)$ as $t \to 0$.

$$
\begin{aligned}
\frac{\epsilon(\omega)}{\epsilon_0} - 1 &= \int_0^\infty dt G(t) e^{i\omega t} = \frac{1}{i\omega} \int_0^\infty dt G(t) \frac{d}{dt} e^{i\omega t} = \frac{iG(0)}{\omega} - \frac{1}{i\omega} \int_0^\infty dt G'(t) e^{i\omega t} \\
&= \frac{iG(0)}{\omega} - \frac{G'(0)}{\omega^2} + \frac{iG''(0)}{\omega^3} + \cdots
\end{aligned} \tag{588}
$$

It is reasonable to assume a smooth turn-on of G at $t = 0$ in which case the first term is absent and then $\epsilon \to \epsilon_0 + O(\omega^{-2})$. at large ω. This is easily fast enough that the contribution from the semicircle at infinity to the Cauchy integral vanishes and we have

$$
\frac{\epsilon(\omega)}{\epsilon_0} - 1 = \int_{-\infty}^\infty \frac{d\omega'}{2\pi i(\omega' - \omega)} \left[\frac{\epsilon(\omega')}{\epsilon_0} - 1 \right], \qquad \text{Im}\,\omega > 0 \tag{589}
$$

We want to take ω to the real axis, but we have to be careful about the resulting singularity in the integrand at $\omega' = \omega$. If there is finite conductivity there is also a pole in $\epsilon(\omega')$ at $\omega' = 0$, $\epsilon \sim i\sigma/\omega$. This pole was not enclosed by the original Cauchy contour C, so when we take the contour to the real axis, it must pass above the conductivity pole. We can handle these singularities by deforming the contour near $\omega' = \omega$ into a semicircle of radius δ about ω into the lower half plane (lhp). Similarly we deform the contour near $\omega' = 0$ into a semicircle of radius δ about 0 into the upper half plane (uhp). Then the contour follows the real axis from $-\infty$ to $-\delta$, a semicircle in the uhp from $-\delta$ to $+\delta$, the real axis from $+\delta$ to $\omega - \delta$, a semicircle in the lhp from $\omega - \delta$ to $\omega + \delta$ and then, finally, the real axis from $\omega + \delta$ to $+\infty$. The real axis contribution, in the limit $\delta \to 0$ is what is traditionally called the Principal Part. The semicircle contributions are just half the full residues of the respective poles. The residue of the pole at 0 contributes negatively because the contour in that case is clockwise.

$$\frac{\epsilon(\omega)}{\epsilon_0} - 1 = P\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i(\omega' - \omega)} \left[\frac{\epsilon(\omega')}{\epsilon_0} - 1\right] + \frac{1}{2}\left(\frac{\epsilon(\omega)}{\epsilon_0} - 1\right) + \frac{i\sigma}{2\epsilon_0\omega} \tag{590}$$

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{i\sigma}{\epsilon_0\omega} + P\int_{-\infty}^{\infty} \frac{d\omega'}{\pi i(\omega' - \omega)} \left[\frac{\epsilon(\omega')}{\epsilon_0} - 1\right] \tag{591}$$

Taking the real and imaginary parts of both sides leads to

$$\mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0} = 1 + P\int_{-\infty}^{\infty} \frac{d\omega'}{\pi(\omega' - \omega)} \mathrm{Im}\,\frac{\epsilon(\omega')}{\epsilon_0} \tag{592}$$

$$\mathrm{Im}\,\frac{\epsilon(\omega)}{\epsilon_0} = \frac{\sigma}{\epsilon_0\omega} - P\int_{-\infty}^{\infty} \frac{d\omega'}{\pi(\omega' - \omega)} \mathrm{Re}\,\left[\frac{\epsilon(\omega')}{\epsilon_0} - 1\right] \tag{593}$$

These *dispersion relations* are also known as Kramers-Kronig relations after the physicists who first introduced them. They reflect the consequences of a few very basic and fundamental physical principles.

For real ω, it follows from the reality of $G(t)$ that $\epsilon(\omega)^* = \epsilon(-\omega)$. Hence Re ϵ is even in ω, whereas Im ϵ is odd. Thus one can write the dispersion integrals with range $0 < \omega < \infty$.

$$\mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0} = 1 + P\int_{0}^{\infty} \frac{d\omega'}{\pi} \frac{2\omega'}{\omega'^2 - \omega^2} \mathrm{Im}\,\frac{\epsilon(\omega')}{\epsilon_0} \tag{594}$$

$$\mathrm{Im}\,\frac{\epsilon(\omega)}{\epsilon_0} = \frac{\sigma}{\epsilon_0\omega} - P\int_{0}^{\infty} \frac{d\omega'}{\pi} \frac{2\omega}{\omega'^2 - \omega^2} \mathrm{Re}\,\left[\frac{\epsilon(\omega')}{\epsilon_0} - 1\right] \tag{595}$$

9.6 Causal Propagation

To set up a test of causal wave propagation, we imagine a wave with a sharp wave front timed to arrive, at normal incidence, at the surface of a dispersive material, located at $x = 0$, at time $t = 0$. For $t < 0$, the wave is entirely in the region $x < 0$ and can be written

$$\text{Incident wave}: \quad \psi(x,t) = \int d\omega A(\omega) e^{i\omega x/c - i\omega t}, \qquad x < 0. \tag{596}$$

$$A(\omega) = \int dt \psi(0,t) e^{i\omega t} \tag{597}$$

Because $\psi(0,t) = 0$ for $t < 0$, $A(\omega)$ can be taken to be analytic in the upper half ω plane.

For $x > 0$ we have, from the normal incidence refraction formula $A_t(\omega) = 2A(\omega)/(1 + n(\omega))$,

$$\text{Refracted wave :} \qquad \psi(x,t) = \int d\omega \frac{2}{1+n(\omega)} A(\omega) e^{i\omega n(\omega)x/c - i\omega t}, \qquad x > 0 \tag{598}$$

The analyticity in the uhp of A, ϵ and n, means that the integrand for $x > 0$ is analytic in the uhp. If the integrand vanishes rapidly enough as $\omega \to \infty$, we can close the contour. To decide when we can do this we need the fact that $n \to 1$ as $\omega \to \infty$, so that $e^{i\omega n(\omega)x/c - i\omega t} \to e^{i\omega(x/c-t)}$. Thus we can close the contour in the uhp when $x > ct$. Then analyticity in the uhp implies, via Cauchy's theorem, that $\psi(x,t) = 0$ for $t < x/c$. That is, no signal can propagate faster than light.

10 Waveguides and Cavities

10.1 The approximation of perfect conductors

In our study of waveguides and cavities, we shall make the simplifying assumption of perfect conductivity. That is time varying fields ($\omega \neq 0$) are strictly zero in the body of the conductor and surface currents and charge densities are instantly adjusted to maintain zero fields when fields just outside the conductor are non-zero. Then Maxwell's equations imply for the fields just outside the conductor the boundary conditions

$$\boldsymbol{n}\cdot\boldsymbol{B}=0, \qquad \boldsymbol{E}_\parallel=0, \qquad \boldsymbol{n}\cdot\boldsymbol{D}=\Sigma, \qquad \boldsymbol{n}\times\boldsymbol{H}_\parallel=\boldsymbol{K} \tag{599}$$

where Σ is the free surface charge density and $\boldsymbol{K}$ is the free surface current density.

To appreciate when this assumption is valid we briefly recall the behavior of time-varying fields near the boundary of conductors with large but finite σ in the quasistatic approximation, where the displacement current is neglected. Then, with $\boldsymbol{J}=\sigma\boldsymbol{E}$, the Maxwell equations read (assuming time dependence $e^{-i\omega t}$ in all fields)

$$\nabla\times\boldsymbol{B} = \mu\sigma\boldsymbol{E}, \quad \nabla\times\boldsymbol{E}=i\omega\boldsymbol{B}, \qquad \nabla\cdot\boldsymbol{B}=0, \quad \nabla\cdot\boldsymbol{E}=0 \tag{600}$$

$$-\nabla^2\boldsymbol{B} = i\mu\sigma\omega\boldsymbol{B}\equiv\frac{2i}{\delta^2}\boldsymbol{B}=-\left(\frac{1-i}{\delta}\right)^2\boldsymbol{B} \tag{601}$$

Suppose we have a non-zero magnetic field parallel to the surface just outside the conductor. Then instead of falling abruptly to zero just inside the conductor, the field drops off exponentially $\boldsymbol{B}(\xi)=\boldsymbol{B}(0)e^{-(1-i)\xi/\delta}$, where ξ is a coordinate that measures the normal depth into the conductor. The skin depth $\delta=\sqrt{2/\mu\sigma\omega}$, which measure how deeply the fields penetrate, goes to zero as $\sigma\to\infty$. Here we are assuming that over a scale δ the fields are essentially constant in the directions parallel to the surface of the conductor. In this situation the corresponding electric field and current density are

$$\begin{aligned}\boldsymbol{E} &= \frac{1}{\mu\sigma}\nabla\times\boldsymbol{B}=\frac{1-i}{\mu\sigma\delta}\boldsymbol{n}\times\boldsymbol{B}(0)e^{-(1-i)\xi/\delta}\\ &= \sqrt{\frac{\omega}{2\mu\sigma}}(1-i)\boldsymbol{n}\times\boldsymbol{B}(0)e^{-(1-i)\xi/\delta}\end{aligned} \tag{602}$$

$$\boldsymbol{J} = \sqrt{\frac{\omega\sigma}{2\mu}}(1-i)\boldsymbol{n}\times\boldsymbol{B}(0)e^{-(1-i)\xi/\delta}, \qquad \boldsymbol{K}=\int_0^\infty d\xi\boldsymbol{J}=\boldsymbol{n}\times\frac{\boldsymbol{B}(0)}{\mu} \tag{603}$$

where $\boldsymbol{n}$ is the outward normal to the conductor. Notice that at $\xi=0$ the electric field is tangential to the surface and small. (For a perfect conductor the tangential component of $\boldsymbol{E}$ would be 0.) This surface tangential field is simply proportional to the effective surface current

$$\boldsymbol{E}_\parallel=\sqrt{\frac{\omega\mu}{2\sigma}}(1-i)\boldsymbol{K}=\frac{1-i}{\sigma\delta}\boldsymbol{K} \tag{604}$$

The proportionality constant has the units of resistance and is called the surface impedance $Z_s = (1-i)/\sigma\delta$. Similarly Faraday's law shows that there would be a small normal component to $\boldsymbol{B}$ at the surface.

The last expression for the effective surface current matches that for the boundary condition at a perfect conductor. We see that the effect of finite conductivity is to spread out the abrupt transition for a perfect conductor to a smooth but rapid transition over a distance of the skin depth. Finite conductivity is also responsible for Ohmic losses. These can be evaluated either via the time averaged Poynting vector at the surface

$$\frac{\text{Power}}{\text{Area}} = \boldsymbol{n}\cdot\langle\boldsymbol{S}\rangle = \frac{1}{2\mu}\boldsymbol{n}\cdot\text{Re }\boldsymbol{E}\times\boldsymbol{B}^*$$

$$= \frac{1}{2\mu}\sqrt{\frac{\omega}{2\mu\sigma}}\boldsymbol{n}\cdot\text{Re }(1-i)(\boldsymbol{n}\times\boldsymbol{B}(0))\times\boldsymbol{B}^*(0) \tag{605}$$

$$= -\frac{\omega\delta}{4\mu}|\boldsymbol{B}(0)|^2 = -\frac{\omega\mu\delta}{4}|\boldsymbol{H}(0)|^2 = -\frac{1}{2\sigma\delta}|\boldsymbol{H}(0)|^2 \tag{606}$$

or as the integral $\int d\xi \text{Re } \boldsymbol{J}\cdot\boldsymbol{E}^*/2$. These losses go to 0 as $\sigma\to\infty$. The advantage of the last expression is that $\boldsymbol{H}$ is continuous across the interface and so in the limit of high conductivity it may be taken as its value just outside the conductor.

10.2 Waveguides

A typical waveguide is a long hollow conductor within which electromagnetic waves can propagate. The fields satisfy Maxwell's equations in the nonconducting hollow region. The effect of the conducting wave guide is, in the limit of a perfect conductor, just the imposition of boundary conditions as discussed in the previous section. We first consider a long cylinder with uniform cross section. The interior of the cylinder is filled with a material with uniform ϵ,μ. We also restrict attention to a perfect conductor, so the boundary conditions on the inner cylindrical wall are $\boldsymbol{E}_\parallel = 0$ and $\boldsymbol{n}\cdot\boldsymbol{B} = 0$. Maxwell's equations imply that each component of $\boldsymbol{E},\boldsymbol{B}$ satisfies the wave equation:

$$\left(-\nabla^2+\epsilon\mu\frac{\partial^2}{\partial t^2}\right)\boldsymbol{E} = 0, \qquad \left(-\nabla^2+\epsilon\mu\frac{\partial^2}{\partial t^2}\right)\boldsymbol{B} = 0 \tag{607}$$

but they also imply relations among those components. For harmonic time variation $e^{-i\omega t}$, Faraday's and the Ampere-Maxwell equations become

$$\boldsymbol{B} = \frac{1}{i\omega}\nabla\times\boldsymbol{E}, \qquad \boldsymbol{E} = -\frac{1}{i\omega\mu\epsilon}\nabla\times\boldsymbol{B} \tag{608}$$

These formula's automatically impose the divergence equations $\nabla\cdot\boldsymbol{E} = \nabla\cdot\boldsymbol{B} = 0$. With cylindrical geometry we choose the z-axis parallel to the interior walls of the cylinder. The walls of the cylinder projected onto the xy-plane will be a closed curve, which for the moment we allow to be arbitrary in shape. Because of this generality we cannot find a plane wave solution in all three coordinates, but we can find solutions with plane wave z dependence $e^{ikz-i\omega t}$. In effect we can find a 1D plane wave solution. So we assume the form $f(x,y)e^{ikz-i\omega t}$ where f is any of the field components. We write $\nabla = \nabla_\perp + \hat{z}\partial/\partial z$, and also $\boldsymbol{E} = E_z\hat{z} + \boldsymbol{E}_\perp$, and $\boldsymbol{B} = B_z\hat{z} + \boldsymbol{B}_\perp$. Then we write out the curl and take its scalar and vector product with $\hat{z}$:

$$\begin{aligned}
\nabla\times\boldsymbol{E} &= \nabla_\perp\times\boldsymbol{E}_\perp + \nabla_\perp E_z\times\hat{z} + ik\hat{z}\times\boldsymbol{E}_\perp && (609)\\
\hat{z}\cdot\nabla\times\boldsymbol{E} &= \hat{z}\cdot\nabla_\perp\times\boldsymbol{E}_\perp && (610)\\
\hat{z}\times(\nabla\times\boldsymbol{E}) &= \hat{z}\times(\nabla_\perp E_z\times\hat{z}) + ik\hat{z}\times(\hat{z}\times\boldsymbol{E}_\perp) = \nabla_\perp E_z - ik\boldsymbol{E}_\perp && (611)\\
\hat{z}\cdot\nabla\times\boldsymbol{B} &= \hat{z}\cdot\nabla_\perp\times\boldsymbol{B}_\perp, \qquad \hat{z}\times(\nabla\times\boldsymbol{B}) = \nabla_\perp B_z - ik\boldsymbol{B}_\perp && (612)
\end{aligned}$$

Employing these decompositions in the Maxwell equations then gives

$$\begin{aligned}
B_z &= \frac{1}{i\omega}\hat{z}\cdot\nabla_\perp\times\boldsymbol{E}_\perp, \qquad E_z = -\frac{1}{i\omega\mu\epsilon}\hat{z}\cdot\nabla_\perp\times\boldsymbol{B}_\perp && (613)\\
\hat{z}\times\boldsymbol{B}_\perp &= \frac{1}{i\omega}(\nabla_\perp E_z - ik\boldsymbol{E}_\perp), \qquad \hat{z}\times\boldsymbol{E}_\perp = -\frac{1}{i\omega\mu\epsilon}(\nabla_\perp B_z - ik\boldsymbol{B}_\perp) && (614)\\
\nabla_\perp\cdot\boldsymbol{E}_\perp + ikE_z &= 0, \qquad \nabla_\perp\cdot\boldsymbol{B}_\perp + ikB_z = 0 && (615)
\end{aligned}$$

The equations on the middle line can be rearranged by taking the cross product of say the first one with $\hat{z}$ as

$$\begin{aligned}
-\boldsymbol{B}_\perp + \frac{k}{\omega}\hat{z}\times\boldsymbol{E}_\perp &= \frac{1}{i\omega}\hat{z}\times\nabla_\perp E_z\\
-\frac{k}{\omega\mu\epsilon}\boldsymbol{B}_\perp + \hat{z}\times\boldsymbol{E}_\perp &= -\frac{1}{i\omega\mu\epsilon}\nabla_\perp B_z && (616)\\
\boldsymbol{B}_\perp\left(\frac{k^2}{\omega^2\mu\epsilon}-1\right) &= \frac{1}{i\omega}\hat{z}\times\nabla_\perp E_z + \frac{k}{i\omega^2\mu\epsilon}\nabla_\perp B_z && (617)\\
\hat{z}\times\boldsymbol{E}_\perp\left(\frac{k^2}{\omega^2\mu\epsilon}-1\right) &= \frac{k}{i\omega^2\mu\epsilon}\hat{z}\times\nabla_\perp E_z + \frac{1}{i\omega\mu\epsilon}\nabla_\perp B_z && (618)
\end{aligned}$$

which shows that, except in the case $k = \pm\omega\sqrt{\mu\epsilon}$, we can express $\boldsymbol{E}_\perp, \boldsymbol{B}_\perp$ in terms of E_z, B_z. **TEM Modes** That exceptional case implies that all components of $\boldsymbol{E}$ and $\boldsymbol{B}$ satisfy the 2D Laplace equation. Since E_z is parallel to the waveguide walls, the perfect conductor boundary conditions require $E_z = 0$ on all boundaries so the uniqueness theorem of the Laplace equation implies $E_z = 0$ everywhere inside the guide. Then the above equations imply

$$E_z = 0, \qquad \boldsymbol{B}_\perp = \frac{k}{\omega}\hat{z}\times\boldsymbol{E}_\perp, \qquad \nabla_\perp B_z = 0, \qquad \text{for } k^2 = \omega^2\mu\epsilon \tag{619}$$

The third equation shows that B_z is independent of x, y. This constant is not directly restricted by the perfect conductor boundary condition $\boldsymbol{n}\cdot\boldsymbol{B} = 0$ since B_z is a parallel component of $\boldsymbol{B}$. However $0 = \nabla\cdot\boldsymbol{B} = \nabla_\perp\cdot\boldsymbol{B}_\perp + ikB_z$ show by Gauss' theorem that

$$\int dxdy\ B_z = \frac{i}{k}\oint dl\ \boldsymbol{n}\cdot\boldsymbol{B}_\perp = 0 \tag{620}$$

by the perfect conductor boundary condition on $\boldsymbol{B}_\perp$ [7]. Then B_z, being a constant, must in fact be 0. Thus the exceptional solution is purely transverse ($E_z = B_z = 0$) and so is called the TEM (Transverse Electric Magnetic) mode. In order for it to exist, the electric field must satisfy the 2D electrostatic equations $\nabla_\perp \cdot \boldsymbol{E}_\perp = 0$, $\nabla_\perp \times \boldsymbol{E}_\perp = 0$, which is equivalent to $\nabla_\perp^2 \phi = 0$, with the boundary condition $\boldsymbol{E}_\parallel = 0$ equivalent to $\phi =$constant on the boundary. There is no such solution for a simply connected 2D region, so it can only be relevant to a situation at least as complicated as a coaxial waveguide.

A simple example waveguide that supports a TEM mode is a pair of concentric conducting cylinders of radii $a < b$ with the gap between them a nonconducting material. Then the 2D electrostatics problem is solved with a radial electric field $\boldsymbol{E}(\rho) = \hat{\rho} E_0 a/\rho$. Then $\boldsymbol{B} = \sqrt{\mu\epsilon}\hat{\varphi} E_0 a/\rho$, and the perfectly conducting boundary conditions $E_\parallel = B_n = 0$ are automatically satisfied. This example is explored further in a homework problem, in which the effects of finite conductivity are also considered.

TE and TM modes: Returning to the general case $k^2 \neq \omega^2\mu\epsilon$ we see that we can describe the general solution in terms of two distinct modes: $E_z = 0$ (TE modes) and $B_z = 0$ (TM modes). TE mode solutions are

$$\boldsymbol{B}_\perp = \frac{ik}{\omega^2\mu\epsilon - k^2}\nabla_\perp B_z = \frac{ik}{\gamma^2}\nabla_\perp B_z \quad (622)$$

$$\hat{z} \times \boldsymbol{E}_\perp = \frac{i\omega}{\omega^2\mu\epsilon - k^2}\nabla_\perp B_z = \frac{i\omega}{\gamma^2}\nabla_\perp B_z = \frac{\omega}{k}\boldsymbol{B}_\perp = \frac{\omega\mu}{k}\boldsymbol{H}_\perp \quad (623)$$

$$(-\nabla_\perp^2 + k^2 - \omega^2\mu\epsilon)B_z = 0, \qquad \boldsymbol{n} \cdot \nabla_\perp B_z = 0 \quad \text{on boundaries} \quad (624)$$

In this case the Neumann boundary conditions on B_z are equivalent to the $B_n = E_\parallel = 0$, as can be easily seen by dotting the first two lines with $\boldsymbol{n}$.

Similarly the TM mode solutions are

$$\boldsymbol{B}_\perp = \frac{i\omega\mu\epsilon}{\omega^2\mu\epsilon - k^2}\hat{z} \times \nabla_\perp E_z = \frac{i\omega\mu\epsilon}{\gamma^2}\hat{z} \times \nabla_\perp E_z \quad (625)$$

$$\hat{z} \times \boldsymbol{E}_\perp = \frac{ik}{\omega^2\mu\epsilon - k^2}\hat{z} \times \nabla_\perp E_z = \frac{ik}{\gamma^2}\hat{z} \times \nabla_\perp E_z = \frac{k}{\omega\epsilon}\boldsymbol{H}_\perp \quad (626)$$

$$(-\nabla_\perp^2 + k^2 - \omega^2\mu\epsilon)E_z = 0, \qquad E_z = 0 \quad \text{on boundaries} \quad (627)$$

Here Dirichlet conditions are necessary since E_z is parallel to the conducting surface. But this is also sufficient since it implies $\mathbf{n} \times \nabla_\perp E_z = 0$. Then dotting the first two equations with $\boldsymbol{n}$ shows that $B_n = E_\parallel = 0$ on the boundaries, as required by perfect conductor boundary conditions. In both TM and TE cases we have to solve the Helmholtz equation, but with Neumann or Dirichlet conditions in TE and TM cases respectively.

[7] Alternatively we can write

$$\int dxdy\ B_z = \frac{1}{i\omega}\int dxdy \hat{z} \cdot (\nabla \times \boldsymbol{E}) = \frac{1}{i\omega}\oint d\boldsymbol{l} \cdot \boldsymbol{E} = 0. \quad (621)$$

We can regard the Helmholtz equation as an eigenvalue problem for the transverse Laplacian

$$-\nabla_\perp^2\psi_a = \gamma_a^2\psi_a, \qquad \gamma^2 = \omega^2\mu\epsilon - k^2 \tag{628}$$

where γ_a^2 are the possible eigenvalues. Since the eigenvalues are a discrete set of numbers, fixed for given geometry of waveguide and boundary conditions, we can interpret $\omega_a(k) = \sqrt{k^2+\gamma_a^2}/\sqrt{\mu\epsilon}$ as the dispersion formula for 1D waves along the waveguide. The group velocity of these waves is then $v_g = k/(\omega_a\mu\epsilon)$.

10.3 Rectangular Waveguide

Let the cross section be a rectangle of dimensions $a \times b$ with $a > b$. The eigenfunctions are

$$\text{TE}: \qquad B^z_{mn} = B_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}, \qquad \gamma_{mn} = \pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} \tag{629}$$

$$\text{TM}: \qquad E^z_{mn} = E_0 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \qquad \gamma_{mn} = \pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} \tag{630}$$

The cutoff frequencies of TE and TM modes have a large overlap. The difference is that the modes with m or n equal to 0 are possible for TE but not TM. In particular the lowest cutoff frequency is the TE mode $\gamma_{10}/\sqrt{\mu\epsilon} = \pi/(a\sqrt{\mu\epsilon})$. For this mode $E_z = B_y = E_x = 0$ and

$$B_z = B_0\cos\frac{\pi x}{a}, \qquad B_x = -\frac{ik}{\gamma^2}\frac{\pi}{a}B_0\sin\frac{\pi x}{a}, \qquad E_y = -\frac{\omega}{k}B_x = \frac{i\omega}{\gamma^2}\frac{\pi}{a}B_0\sin\frac{\pi x}{a} \tag{631}$$

10.4 Energy Flow and Attenuation

To calculate the energy and energy flow in a wave guide we use the formulas for the time-averaged energy density and Poynting vectors valid for harmonic time dependence $e^{-i\omega t}$:

$$\langle u\rangle = \frac{1}{2}\text{Re}\left(\frac{\epsilon}{2}\boldsymbol{E}\cdot\boldsymbol{E}^* + \frac{\mu}{2}\boldsymbol{H}\cdot\boldsymbol{H}^*\right), \qquad \langle\boldsymbol{S}\rangle = \frac{1}{2}\text{Re}\boldsymbol{E}\times\boldsymbol{H}^* \tag{632}$$

We first evaluate these expressions assuming perfect conductivity. The evaluation is slightly different for TE and TM modes.
TE Modes For the former

$$\begin{aligned}
\langle\boldsymbol{S}_{\text{TE}}\rangle &= \frac{1}{2}\text{Re}\left[\boldsymbol{E}_\perp\times\boldsymbol{H}_\perp^* + \boldsymbol{E}_\perp\times\hat{z}H_z^*\right] \\
&= \frac{1}{2\mu}\text{Re}\left[\frac{k}{\omega}\boldsymbol{E}_\perp\times(\hat{z}\times\boldsymbol{E}_\perp^*) + \boldsymbol{E}_\perp\times\hat{z}B_z^*\right] \\
&= \frac{1}{2\mu}\text{Re}\left[\hat{z}\frac{k}{\omega}\boldsymbol{E}_\perp\cdot\boldsymbol{E}_\perp^* + \frac{i\omega}{k^2-\omega^2\mu\epsilon}B_z^*\nabla_\perp B_z\right] \\
&= \frac{1}{2\mu}\text{Re}\left[\hat{z}\frac{\omega k}{(k^2-\omega^2\mu\epsilon)^2}\nabla_\perp B_z\cdot\nabla_\perp B_z^* + \frac{i\omega}{k^2-\omega^2\mu\epsilon}B_z^*\nabla_\perp B_z\right]
\end{aligned}$$

The total power transported along the guide is

$$\begin{aligned}
P_{\rm TE} &= \int dxdy\hat{z}\cdot\langle \boldsymbol{S}\rangle \\
&= \frac{1}{2\mu}\frac{\omega k}{(k^2-\omega^2\mu\epsilon)^2}\int d^2x\nabla_\perp B_z\cdot\nabla_\perp B_z^* = \frac{1}{2\mu}\frac{\omega k}{\gamma^2}\int d^2x|B_z|^2 \qquad (633)
\end{aligned}$$

Finally, the time averaged energy density and energy per unit length are

$$\begin{aligned}
\langle u_{TE}\rangle &= \frac{1}{2}{\rm Re}\left(\frac{\epsilon}{2}\boldsymbol{E}_\perp\cdot\boldsymbol{E}_\perp^* + \frac{1}{2\mu}\boldsymbol{B}_\perp\cdot\boldsymbol{B}_\perp^* + \frac{1}{2\mu}|B_z|^2\right) \\
&= \frac{1}{2}{\rm Re}\left(\frac{\omega^2\epsilon\mu+k^2}{2\mu(\omega^2\mu\epsilon-k^2)^2}\nabla_\perp B_z\cdot\nabla_\perp B_z^* + \frac{1}{2\mu}|B_z|^2\right) \qquad (634)\\
T &\equiv \int d^2x\langle u_{TE}\rangle = \int d^2x\frac{1}{2}\left(\frac{\omega^2\epsilon\mu+k^2}{2\mu(\omega^2\mu\epsilon-k^2)}B_zB_z^* + \frac{1}{2\mu}|B_z|^2\right) \\
&= \frac{1}{2\mu}\frac{\omega^2\epsilon\mu}{\gamma^2}\int d^2x|B_z|^2 = \frac{\omega\mu\epsilon}{k}P_{\rm TE} = \frac{P_{\rm TE}}{v_g} \qquad (635)
\end{aligned}$$

The relation $P_{\rm TE} = v_gT$ precisely corresponds to energy flowing down the guide with the group velocity.

TM Modes The corresponding evaluation for TM modes is similar;

$$\begin{aligned}
\langle \boldsymbol{S}_{\rm TM}\rangle &= \frac{1}{2}{\rm Re}\left[\boldsymbol{E}_\perp\times\boldsymbol{H}_\perp^* + \boldsymbol{E}_z\hat{z}\times\boldsymbol{H}_\perp^*\right] \\
&= \frac{1}{2\mu}{\rm Re}\left[\frac{\omega\mu\epsilon}{k}\boldsymbol{E}_\perp\times(\hat{z}\times\boldsymbol{E}_\perp^*) + \boldsymbol{E}_z\hat{z}\times\boldsymbol{B}_\perp^*\right] \\
&= \frac{1}{2\mu}{\rm Re}\left[\hat{z}\frac{\omega\mu\epsilon}{k}\boldsymbol{E}_\perp\cdot\boldsymbol{E}_\perp^* + \frac{i\omega\mu\epsilon}{k^2-\omega^2\mu\epsilon}E_z\nabla_\perp E_z^*\right] \\
&= \frac{\epsilon}{2}{\rm Re}\left[\hat{z}\frac{\omega k}{(k^2-\omega^2\mu\epsilon)^2}\nabla_\perp E_z\cdot\nabla_\perp E_z^* + \frac{i\omega}{k^2-\omega^2\mu\epsilon}E_z\nabla_\perp E_z^*\right]
\end{aligned}$$

The total power transported along the guide is

$$\begin{aligned}
P_{\rm TM} &= \int dxdy\hat{z}\cdot\langle \boldsymbol{S}\rangle \\
&= \frac{\epsilon}{2}\frac{\omega k}{(k^2-\omega^2\mu\epsilon)^2}\int d^2x\nabla_\perp E_z\cdot\nabla_\perp E_z^* = \frac{\epsilon}{2}\frac{\omega k}{\gamma^2}\int d^2x|E_z|^2 \qquad (636)
\end{aligned}$$

Finally, the time averaged energy density and energy per unit length are

$$\begin{aligned}
\langle u_{TM}\rangle &= \frac{1}{2}{\rm Re}\left(\frac{\epsilon}{2}\boldsymbol{E}_\perp\cdot\boldsymbol{E}_\perp^* + \frac{1}{2\mu}\boldsymbol{B}_\perp\cdot\boldsymbol{B}_\perp^* + \frac{\epsilon}{2}|E_z|^2\right) \\
&= \frac{1}{2}{\rm Re}\left(\frac{\epsilon(\omega^2\epsilon\mu+k^2)}{2(\omega^2\mu\epsilon-k^2)^2}\nabla_\perp E_z\cdot\nabla_\perp E_z^* + \frac{\epsilon}{2}|E_z|^2\right) \qquad (637)
\end{aligned}$$

$$
\begin{aligned}
T &\equiv \int d^2x\langle u_{TE}\rangle = \int d^2x\frac{\epsilon}{2}\left(\frac{\omega^2\epsilon\mu+k^2}{2(\omega^2\mu\epsilon-k^2)}E_zE_z^*+\frac{1}{2}|E_z|^2\right)\\
&= \frac{\epsilon}{2}\frac{\omega^2\epsilon\mu}{\gamma^2}\int d^2x|E_z|^2 = \frac{\omega\mu\epsilon}{k}P_{\rm TM} = \frac{P_{\rm TM}}{v_g} \qquad (638)
\end{aligned}
$$

Resistive Losses Now that we have the solutions for perfect conductivity we can use them to get a first order estimate of the losses that occur with large but finite conductivity. The Ohmic losses are concentrated within a depth δ from the surface of the conductor and the loss per unit area was found in Eq.(**??**) to be $\omega\delta\boldsymbol{B}_\parallel\cdot\boldsymbol{B}_\parallel^*/4\mu_c = |\boldsymbol{H}_\parallel|^2/2\sigma\delta$. We express the result in terms of $\boldsymbol{H}_\parallel$ because that field is continuous across the interface. Then the only properties of the conductor that come in are δ,σ. The loss per unit length is the integral of this quantity along a ribbon of width dz that encircles the waveguide. For the TM modes the parallel component of the magnetic field just outside the conductor's surface is approximately the value for perfect conductivity and can be obtained from the cross product
TM Modes:

$$
\begin{aligned}
\boldsymbol{n}\times\boldsymbol{H}_\perp &= \frac{-i\omega\epsilon}{k^2-\omega^2\mu\epsilon}\boldsymbol{n}\times(\hat{z}\times\nabla_\perp E_z) = \frac{-i\omega\epsilon}{k^2-\omega^2\mu\epsilon}(\hat{z}\boldsymbol{n}\cdot\nabla_\perp E_z) \qquad (639)\\
\frac{dP_{\rm TM}}{dz} &= -\frac{1}{2\sigma\delta}\frac{\omega^2\epsilon^2}{(k^2-\omega^2\mu\epsilon)^2}\oint dl|\boldsymbol{n}\cdot\nabla_\perp E_z|^2\\
&= -\frac{1}{2\sigma\delta}\frac{\omega^2\epsilon^2}{\gamma^4}\oint dl|\boldsymbol{n}\cdot\nabla_\perp E_z|^2 \qquad (640)
\end{aligned}
$$

TE Modes For TE modes, there are two contributions to the parallel magnetic field: H_z and the parallel component of $\boldsymbol{H}_\perp$ which can be inferred from

$$
\begin{aligned}
\boldsymbol{n}\times\boldsymbol{H}_\perp &= \frac{-ik}{k^2-\omega^2\epsilon}\boldsymbol{n}\times\nabla_\perp H_z \qquad (641)\\
\frac{dP_{\rm TE}}{dz} &= -\frac{1}{2\sigma\delta}\oint dl\left(H_z^2+\frac{k^2}{(k^2-\omega^2\mu\epsilon)^2}|\boldsymbol{n}\times\nabla_\perp H_z|^2\right)\\
&= -\frac{1}{2\sigma\delta}\oint dl\left(H_z^2+\frac{k^2}{\gamma^4}|\boldsymbol{n}\times\nabla_\perp H_z|^2\right) \qquad (642)
\end{aligned}
$$

The **attenuation constant** β is defined according to an exponential law $P(z) = P(0)e^{-2\beta z}$, so that

$$
\begin{aligned}
\beta &= -\frac{1}{2P}\frac{dP}{dz} \qquad (643)\\
\beta_{\rm TE} &= \frac{\gamma^2}{2\sigma\delta}\left[\frac{\oint dlH_z^2+(k^2/\gamma^4)\oint dl|\boldsymbol{n}\times\nabla_\perp H_z|^2}{\mu\omega k\int d^2x|H_z|^2}\right] \qquad (644)\\
\beta_{\rm TM} &= \frac{1}{2\sigma\delta}\frac{\omega\epsilon}{\gamma^2 k}\frac{\oint dl|\boldsymbol{n}\cdot\nabla_\perp E_z|^2}{\int d^2x|E_z|^2} \qquad (645)
\end{aligned}
$$

Since the integrals in these expressions depend only on the geometry of the wave guide, the frequency dependence of the β's is explicit (remembering that $\delta\sim\omega^{-1/2}$) and assuming that

$\mu, \mu_c, \epsilon, \sigma$ are all frequency independent). Then $\beta_{\rm TM}$ is proportional to

$$\frac{\omega}{\delta k} = \frac{1}{\sqrt{\mu\epsilon}\delta_0}\sqrt{\frac{\omega}{\omega_0}}\left(1-\frac{\omega_0^2}{\omega^2}\right)^{-1/2} \tag{646}$$

where we have written $\omega_0 = \gamma/\sqrt{\mu\epsilon}$ for the cutoff frequency and δ_0 for the skin depth at cutoff frequency. This expression blows up at the cutoff frequency, grows as $\sqrt{\omega}$ at high frequency, and has a minimum at $\omega = \omega_0\sqrt{3}$. The frequency dependence of $\beta_{\rm TE}$ is more involved because of the two contributions:

$$\beta_{\rm TE} = \sqrt{\frac{\omega}{\omega_0}}\left(1-\frac{\omega_0^2}{\omega^2}\right)^{-1/2}\left[I_1\frac{\omega_0^2}{\omega^2} + I_2\left(1-\frac{\omega_0^2}{\omega^2}\right)\right] \tag{647}$$

It also blows up at cutoff and increases as $\sqrt{\omega}$ at high frequency, but its minimum depends on the details of the wave guide.

10.5 Resonant Cavities

Waveguides, because they leave one direction (z in our discussion) unrestricted, allow waves to propagate in that direction. The walls of the guide prevent propagation in the x and y directions. These waves were described mathematically by assuming the dependence $f(x,y)e^{ikz-i\omega t}$ for the various components of the fields and then finding solutions for a continuous range of $kc = \sqrt{\omega^2-\omega_c^2}$. Except for TEM modes, the confinement of fields in the xy dimensions generally produces a discrete set of lower cutoffs $\omega_c > 0$ on the allowed frequency of propagation. It is instructive to invert the relationship between ω and k: $\omega = c\sqrt{k^2+\gamma^2}$, which is reminiscent of the relativistic relation of energy to momentum. If we compare this to the relation for 3D space $\omega = c\sqrt{k_x^2+k_y^2+k_z^2}$, we can interpret the effect of the waveguide walls as quantizing the possible values of $k_x^2+k_y^2$. Similarly, we could have set up a guide that allowed propagation in two directions by considering waves in the region between two parallel planes, say at $x=0$ and $x=L$. Then the wave ansatz would be $f(x)e^{ik_yy+ik_zz-i\omega t}$, and we would obtain a dispersion law $\omega = c\sqrt{k_y^2+k_z^2+\gamma^2}$, where now the effect of the walls is just to quantize the values of k_x. Notice that the effect of confining motion in one dimension has the effect of dimensional reduction. The extra dimension is not completely lost but makes itself felt through the spectrum of "masses" represented by the cutoff frequencies. If the extra dimension is very small, it would take a huge energy ($\hbar\omega$) to excite these extra modes. Some very speculative notions about high energy physics include the hypothesis that our space-time actually has extra dimensions that are extremely tiny, that we will only discover in extremely high energy experiments by measuring the spectrum of cutoff frequencies, which to us will be a spectrum of particle masses.

Returning to reality, we now consider confinement in all three dimensions, that is we consider electromagnetic fields in cavities. Then we can expect that the frequency itself will be quantized. Again we simplify life by first assuming perfectly conducting walls, and consider the effect of finite conductivity later as a small perturbation. Since we have already studied cylindrical waveguides, the easiest cavities to discuss are cylinders with conducting end plates at fixed values of z. But of course cavities of any shape could be considered. One of the homework problems asks you to investigate a spherical cavity.

To describe a cylindrical cavity we can start with our known solutions for a cylindrical waveguide and restrict the z dependence to satisfy the appropriate boundary conditions at $z=0$ and $z=L$. We use the fact that for a given frequency, we get a waveguide solution for both signs of $k=\pm\sqrt{\omega^2-\omega_c^2}/c$. Thus each field component will have the general dependence $f(x,y)(A\sin kz+B\cos kz)$. We can maintain the distinction between TM and TE modes. At the endplates we must have $E_{x,y}=0$ and $B_z=0$. For TE modes, which are given in terms of B_z, we see that these boundary conditions imply $B_z=B_z(x,y)\sin(n\pi z/L)$. Because $E_{x,y}$ are determined in terms of $\partial_{x,y}B_z$, those components of $\boldsymbol{E}$ will automatically vanish as well. Thus the frequency will be quantized to the discrete values $\omega=c\sqrt{\gamma^2+n^2\pi^2/L^2}$. The effect of the endplates has been to quantize $k=\pm n\pi/L$. The x,y components of the fields are

$$\begin{aligned}
B_z &= B_z(x,y)\sin\frac{n\pi z}{L}\\
\boldsymbol{B}_\perp &= \frac{i}{\gamma^2}\nabla_\perp B_z(x,y)\left(\frac{n\pi}{2iL}e^{i\pi nz/L}-\frac{-n\pi}{2iL}e^{-i\pi nz/L}\right)=\frac{n\pi}{\gamma^2 L}\nabla_\perp B_z(x,y)\cos\frac{n\pi z}{L}\\
\hat{z}\times\boldsymbol{E}_\perp &= \frac{i\omega}{\gamma^2}\nabla_\perp B_z(x,y)\sin\frac{n\pi z}{L} \qquad \text{TE Modes}
\end{aligned}\tag{648}$$

For TM modes $B_z=0$ and the solution is specified in terms of E_z. For the wave guide solution with z dependence $e^{\pm ikz}$ we had

$$\boldsymbol{E}_\perp=\pm\frac{ik}{\gamma^2}\nabla_\perp E_z e^{\pm ikz}\tag{649}$$

to get a linear combination of these to vanish at $z=0,L$ we clearly need the two terms to appear with the same coefficients in E_z (so they appear with opposite coefficients in $\boldsymbol{E}_\perp$) to get $\boldsymbol{E}_\perp=0$ at $z=0$. Then $\boldsymbol{E}_\perp=0$ at $z=l$ requires $k=n\pi/L$. Thus we have

$$\begin{aligned}
E_z &= E_z(x,y)\cos\frac{n\pi z}{L}, \qquad \boldsymbol{B}_\perp=\frac{i\omega\mu\epsilon}{\gamma^2}\hat{z}\times\nabla_\perp E_z(x,y)\cos\frac{n\pi z}{L}\\
\hat{z}\times\boldsymbol{E}_\perp &= -\frac{n\pi}{\gamma^2 L}\hat{z}\times\nabla_\perp E_z(x,y)\sin\frac{n\pi z}{L} \qquad \text{TM Modes}
\end{aligned}\tag{650}$$

For the **right circular cylinder** the transverse solutions are Bessel functions $J_m(\gamma_{mn}\rho)e^{im\varphi}$. For TM modes Dirichlet conditions on E_z imply that $R\gamma_{mn}\equiv x_{mn}$ the zeroes of $J_m(x)$. Then the cavity frequencies are $\omega_{mnp}=c\sqrt{x_{mn}^2/R^2+p^2\pi^2/L^2}$. The lowest TM frequency is $\omega_{010}=cx_{01}/R\approx 2.405c/R$. For TE modes the Neumann conditions of B_z imply that $R\gamma_{mn}=y_{mn}$ the zeroes of $J_m'(x)$. For TE modes $p=1$ is the lowest allowed z mode, and $y_{11}\approx 1.841$ is the lowest zero of the various J_m'. Thus the lowest TE frequency is $\omega_{111}=c\sqrt{y_{11}^2/R^2+\pi^2/L^2}$. since $y_{11}<x_{01}$, this TE mode will have the lowest frequency for sufficiently large L/R.

Energy, Finite Conductivity, and Q. The energy stored in a cylindrical cavity for a single mode can be obtained from the energy per unit length of a wave guide by multiplying by $\cos^2(n\pi z/L)$ or $\sin^2(nz\pi/L)$ and integrating z from 0 to L, which amounts, for $n \neq 0$ to multiplying the energy per unit length by $L/2$. For the $n = 0$ mode one instead multiplies by L. This gives

$$U_{\rm TE} = \frac{L}{4\mu}\frac{\gamma^2+n^2\pi^2/L^2}{\gamma^2}\int d^2x|B_z|^2 = \frac{\mu L}{4}\left(1+\frac{n^2\pi^2}{\gamma^2L^2}\right)\int d^2x|H_z|^2 \qquad (651)$$

$$U_{\rm TM} = \frac{\epsilon L}{4}\left(1+\frac{n^2\pi^2}{\gamma^2L^2}\right)\int d^2x|E_z|^2,\ n\neq 0; \qquad U_{\rm TM} = \frac{\epsilon L}{2}\int d^2x|E_z|^2,\ n=0 \qquad (652)$$

The rate of energy loss can be obtained by integrating the rate per unit length for a waveguide (times $\cos^2 kz$ or $\sin^2 kz$) over z from 0 to L, and then adding the contribution from the two endplates.

$$\begin{aligned} P_{\rm TM} &= -\frac{1}{2\sigma\delta}\left[\frac{L}{2}\frac{\omega^2\epsilon^2}{\gamma^4}\oint dl|\boldsymbol{n}\cdot\nabla_\perp E_z|^2 + 2\int d^2x|\boldsymbol{H}_\perp|^2\right] \\ &= -\frac{1}{2\sigma\delta}\left[\frac{L}{2}\frac{\omega^2\epsilon^2}{\gamma^4}\oint dl|\boldsymbol{n}\cdot\nabla_\perp E_z|^2 + 2\frac{\omega^2\epsilon^2}{\gamma^4}\int d^2x|\nabla_\perp E_z|^2\right] \\ &= -\frac{\epsilon}{\sigma\delta\mu}\left(1+\frac{n^2\pi^2}{\gamma^2L^2}\right)\left[\frac{L}{4\gamma^2}\oint dl|\boldsymbol{n}\cdot\nabla_\perp E_z|^2 + \int d^2x|E_z|^2\right], \qquad n\neq 0 \qquad (653)\end{aligned}$$

The last term in square brackets is the contribution of the ends, and the first term in square brackets is multiplied by 2 for the case $n = 0$. The quality factor Q of a resonant cavity is defined by

$$Q = \omega\frac{\text{EnergyStored}}{\text{PowerLoss}}. \qquad (654)$$

Writing $C\xi/A = \oint dl|\boldsymbol{n}\cdot\nabla_\perp E_z|^2/\gamma^2\int d^2x|E_z|^2$, with C the circumference and A the cross sectional area of the cylinder, we can write

$$\begin{aligned} Q_{\rm TM} &= \frac{\omega\sigma\mu\delta L}{4(1+\xi CL/4A)} = \frac{\mu L}{2\mu_c\delta(1+\xi CL/4A)}, \qquad n\neq 0 \\ Q_{\rm TM} &= \frac{\omega\sigma\mu\delta L}{2(1+\xi CL/2A)} = \frac{\mu L}{\mu_c\delta(1+\xi CL/2A)}, \qquad n=0. \qquad (655)\end{aligned}$$

For TE modes, there are two contributions to the parallel magnetic field on the cylinder wall, H_z and the parallel component of $\boldsymbol{H}_\perp$, as well as the two endplate contributions:

$$\begin{aligned} P_{\rm TE} &= -\frac{1}{2\sigma\delta}\left[\oint dl\frac{L}{2}\left(H_z^2+\frac{n^2\pi^2}{L^2\gamma^4}|\boldsymbol{n}\times\nabla_\perp H_z|^2\right) + 2\frac{n^2\pi^2}{L^2\gamma^4}\int d^2x|\nabla_\perp H_z|^2\right] \\ &= -\frac{1}{\sigma\delta}\left[\oint dl\frac{L}{4}\left(H_z^2+\frac{n^2\pi^2}{L^2\gamma^4}|\boldsymbol{n}\times\nabla_\perp H_z|^2\right) + \frac{n^2\pi^2}{L^2\gamma^2}\int d^2x|H_z|^2\right] \qquad (656)\end{aligned}$$

The expression for $Q_{\rm TE}$ is more cumbersome because of the extra contribution. There are now two parameters associated with the transverse geometry.

$$\frac{C\zeta}{A} = \frac{\oint dl|\boldsymbol{n}\times\nabla_\perp H_z|^2}{\gamma^2\int d^2x|H_z|^2}, \qquad \frac{C\eta}{A} = \frac{\oint dl|H_z|^2}{\int d^2x|H_z|^2} \qquad (657)$$

Then we can write

$$Q_{\rm TE} = \frac{\omega\sigma\delta\mu L}{4}\frac{L^2\gamma^2+n^2\pi^2}{(L^2\gamma^2\eta+n^2\pi^2\zeta)(LC/4A)+n^2\pi^2}$$
$$= \frac{\mu L}{2\mu_c\delta}\frac{L^2\gamma^2+n^2\pi^2}{(L^2\gamma^2\eta+n^2\pi^2\zeta)(LC/4A)+n^2\pi^2} \qquad (658)$$

The quality factor controls how rapidly an excited cavity loses its energy. From its definition we have

$$\frac{dU}{dt} = -\frac{\omega_0}{Q}U, \qquad U(t)=U(0)e^{\frac{-\omega_0 t}{Q}} \qquad (659)$$

Where we write ω_0 for the eigenfrequency of the mode excited. Such exponential damping corresponds to damping in the fields at half the rate. So the effect of finite conductivity is to replace harmonic time dependence in the fields $e^{-i\omega_0 t}$ by $e^{-\omega_0 t/2Q-i(\omega_0+\Delta\omega)t}$. This time dependence corresponds to a smoothed out frequency profile

$$E(\omega) = \int_0^\infty \frac{dt}{2\pi}e^{i\omega t}E_0 e^{-\omega_0 t/2Q-i(\omega_0+\Delta\omega)t} = \frac{E_0}{2\pi}\frac{1}{\omega_0/2Q+i(\omega_0+\Delta\omega-\omega)} \qquad (660)$$
$$|E(\omega)|^2 = \frac{|E_0|^2}{4\pi^2}\frac{1}{(\omega-\omega_0-\Delta\omega)^2+\omega_0^2/4Q^2} \qquad (661)$$

So instead of showing a single discrete frequency ω_0, damping spreads out the frequency profile in the energy density over a peak with width at half maximum of $\Gamma=\omega_0/Q$. As $Q\to\infty$ the profile sharpens to a delta function centered at ω_0.

10.6 Perturbation of Boundary Conditions

We can get a little more information about the effects of finite conductivity by considering how its presence alters the boundary value problem for the Helmholtz equation. We have seen that the tangential electric field is not strictly zero near the conductor, but rather

$$\boldsymbol{E}_\parallel \approx \frac{1-i}{\sigma\delta}\boldsymbol{K} = \frac{1-i}{\sigma\delta}\boldsymbol{n}\times\boldsymbol{H} \qquad (662)$$

Thus, taking the simpler case of TM modes, we see that instead of $E_z=0$ on the walls of the cylinder, we should have

$$E_z\hat{z} \approx \frac{1-i}{\sigma\delta}\boldsymbol{n}\times\boldsymbol{H}_\perp \approx \hat{z}\frac{1-i}{\sigma\delta}\frac{i\omega\epsilon}{\gamma^2}\boldsymbol{n}\cdot\nabla_\perp E_z$$
$$E_z \approx \frac{1+i}{\sigma\delta}\frac{\omega\epsilon}{\gamma_0^2}\boldsymbol{n}\cdot\nabla_\perp E_{z0} = (1+i)\frac{\omega^2\mu_c\delta\epsilon}{2\gamma_0^2}\boldsymbol{n}\cdot\nabla_\perp E_{z0} \qquad (663)$$

where we added a 0 subscript on quantities valid in the perfect conductor limit. On the end plates instead of $\boldsymbol{E}_\perp=0$, we should have

$$\boldsymbol{E}_\perp \approx \frac{1-i}{\sigma\delta}\pm\hat{z}\times\boldsymbol{H}_\perp \approx \mp\frac{1-i}{\sigma\delta}\frac{i\omega\epsilon}{\gamma_0^2}\nabla_\perp E_{z0} = \mp(1+i)\frac{\omega^2\mu_c\delta\epsilon}{2\gamma_0^2}\nabla_\perp E_{z0}$$
$$\frac{\partial E_z}{\partial z} = -\nabla_\perp\cdot\boldsymbol{E}_\perp \approx \pm(1+i)\frac{\omega^2\mu_c\delta\epsilon}{2\gamma_0^2}\nabla_\perp^2 E_{z0} = \mp(1+i)\frac{\omega^2\mu_c\delta\epsilon}{2}E_{z0} \qquad (664)$$

where the upper (lower) sign applies to the endplate at $z = L$ ($z = 0$). Denoting the perfect conductor solutions with a 0 subscript, we have for infinite and finite conductivity

$$(-\nabla^2 - \epsilon\mu\omega_0^2)E_{z0} = 0, \qquad E_{z0} = 0 \quad \text{on walls}, \quad \frac{\partial E_{z0}}{\partial z} = 0 \quad \text{on endplates} \tag{665}$$

$$\begin{aligned}(-\nabla^2 - \epsilon\mu\omega^2)E_z &= 0, \qquad E_z = (1+i)\frac{\omega^2\mu_c\delta\epsilon}{2\gamma_0^2}\boldsymbol{n}\cdot\nabla_\perp E_{z0} \quad \text{on walls},\\ &\frac{\partial E_z}{\partial z} = \mp(1+i)\frac{\omega^2\mu_c\delta\epsilon}{2}E_{z0} \quad \text{on endplates}\end{aligned} \tag{666}$$

We now use Green's Theorem

$$\begin{aligned}\int dz d^2x(E_z\nabla^2E_{z0}^* - E_{z0}^*\nabla^2E_z) &= \oint dS(-\boldsymbol{n})\cdot(E_z\nabla E_{z0}^* - E_{z0}^*\nabla E_z))\\ \mu\epsilon(\omega^2-\omega_0^2)\int d^3xE_zE_{z0}^* &= \int dz\oint dl(-\boldsymbol{n})\cdot(E_z\nabla E_{z0}^* - E_{z0}^*\nabla E_z))\\ &\quad -\int d^2x\left(E_z\frac{\partial E_{z0}^*}{\partial z} - E_{z0}^*\frac{\partial E_z}{\partial z}\right)\Big|_{z=0}^{z=L}\\ &= \int dz\oint dl(-\boldsymbol{n})\cdot E_z\nabla E_{z0}^* + \int d^2xE_{z0}^*\frac{\partial E_z}{\partial z}\Big|_{z=0}^{z=L}\end{aligned}$$

where $\boldsymbol{n}$ is the normal vector pointing out from the conductor or into the hollow region, and in the last line we used the fact that $E_{z0} = 0$ on the cylinder walls and $\partial E_{z0}/\partial z = 0$ on the endplates. Putting everything together we have

$$\begin{aligned}(\omega^2-\omega_0^2)\int d^3xE_{z0}E_{z0}^* &\approx -(1+i)\frac{\omega_0^2\mu_c\delta}{\mu}\left(\frac{L}{4\gamma_0^2}\oint dl|\boldsymbol{n}\cdot\nabla_\perp E_{z0}|^2 + \int d^2x|E_{z0}|^2\right)\\ \omega-\omega_0 &\approx -(1+i)\frac{\omega_0\mu_c\delta}{L\mu}\left(\frac{\xi LC}{4A}+1\right) = -(1+i)\frac{\omega_0}{2Q}\end{aligned} \tag{667}$$

From the time dependence implied by finite Q, we infer that $\omega - \omega_0 = \Delta\omega - i\omega_0/2Q$ from which we infer that

$$\Delta\omega = -\frac{\omega_0\mu_c\delta}{L\mu}\left(\frac{\xi LC}{4A}+1\right) = -\frac{\omega_0}{2Q}, \qquad Q = \frac{L\mu}{2\delta\mu_c(1+\xi CL/4A)} \tag{668}$$

The result for Q agrees with that obtained from our energy analysis and the result for $\Delta\omega$ is some new information.

10.7 Excitation of Waveguide Modes

One way to set up waves in waveguides is to apply an oscillating current distribution at some spot within the guide. This current would in turn produce fields oscillating at the same frequency, which far from the current source would be in a superposition of the allowed propagating modes at that frequency. That is, the modes with $\gamma^2 < \omega^2\mu\epsilon$. Close to the source all the other modes will be present to form a proper solution of the Maxwell equations in the presence of the source. This process is a mini-radiation problem in which the outgoing waves are limited to two directions $+z$ and $-z$. If we assume the driving source is localized in z, say $J^\mu = 0$ when $|z| > L/2$, then the fields will solve the sourceless Maxwell equations in the whole region to the right $z > L/2$ and in the whole region to the left $z < -L/2$. In both cases the solution should be a superposition of outgoing waves. This means that the fields on the right will be superpositions of wave guide modes with e^{ikz} dependence for real $k = \sqrt{\omega^2\mu\epsilon - \gamma^2}$ or $e^{-z|k|}$ for imaginary k. similarly on the left they will be superpositions of e^{-ikz} or $e^{+z|k|}$.

An effective way to describe this physical situation is to expand the fields in waveguide normal modes. As we have seen these modes divide into TM and TE modes, in which all components of the field are expressed in terms of E_z and H_z respectively. As discussed in J, Problem 8.18, different modes of each type are orthogonal in the sense that

$$\int d^2x E_{z\lambda}E_{z\mu} = 0, \quad \text{or} \quad \int d^2x H_{z\lambda}H_{z\mu} = 0, \qquad \lambda \neq \mu \tag{669}$$

if $\lambda \neq \mu$. For $\lambda = \mu$ the integrals are nonzero and we may establish a convenient normalization for them. Because the Helmholtz equation and boundary conditions are real, the modes could be chosen to be real. However, it will be convenient to allow some modes to be purely imaginary, and for later convenience adopt the normalizations

$$\int d^2x E_{z\lambda}E_{z\mu} = -\frac{\gamma_\lambda^2}{k_\lambda^2}\delta_{\lambda\mu}, \qquad \text{TM Modes} \tag{670}$$

$$\int d^2x H_{z\lambda}H_{z\mu} = -\frac{\gamma_\lambda^2}{k_\lambda^2 Z_\lambda^2}\delta_{\lambda\mu}, \qquad \text{TE Modes} \tag{671}$$

$$Z_\lambda = \frac{\mu\omega}{k} = \sqrt{\frac{\mu}{\epsilon}}\left(\frac{\omega\sqrt{\mu\epsilon}}{k}\right), \quad \lambda \in \text{TE}$$

$$Z_\lambda = \frac{k}{\epsilon\omega} = \sqrt{\frac{\mu}{\epsilon}}\left(\frac{k}{\omega\sqrt{\epsilon\mu}}\right), \quad \lambda \in \text{TM} \tag{672}$$

The *wave impedance* Z_λ (it has dimensions of resistance) is defined so that $\boldsymbol{H}_\perp = Z_\lambda^{-1}\hat{z}\times\boldsymbol{E}_\perp$ for both kinds of modes. We see that this normalization implies that for TM modes E_z is imaginary for real k_λ and E_z is real for imaginary k_λ. For TE modes, however, H_z is imaginary for both cases. Finally note that, with these conventions, the dimensions of the mode functions $E_{z\lambda}$ are taken to be 1/Length, instead of the physical dimensions of an electric field (Force/Charge). This is a convenient choice for their use as expansion functions, and simply means that the physical units of the electric field will be restored by the units of the expansion coefficients. However, the dimensions of $H_{z\lambda}/E_{z\lambda}$ remain 1/Ohms, the same as for physical fields.

The purpose of these strange normalization conditions on the z-components was to arrange simple orthonormality conditions on $\boldsymbol{E}_\perp$:

$$\int d^2x \boldsymbol{E}_{\perp\lambda}\cdot\boldsymbol{E}_{\perp\mu} = \delta_{\lambda\mu}, \quad \text{and} \quad \int d^2x \boldsymbol{H}_{\perp\lambda}\cdot\boldsymbol{H}_{\perp\mu} = \frac{1}{Z_\lambda^2}\delta_{\lambda\mu} \tag{673}$$

It is also easy to check that the transverse fields for TE modes are orthogonal to those for TM modes. Notice that the signs of the right sides of these equations reflect that fact that $\boldsymbol{E}_\perp$ is always real with our conventions, whereas $\boldsymbol{H}_\perp$ is real for propagating modes (k_λ real) but imaginary for nonpropagating modes (k_λ imaginary). Finally the orthogonality relation

$$\int d^2x \boldsymbol{E}_{\perp\lambda} \times \boldsymbol{H}_{\perp\mu} = \frac{1}{Z_\lambda}\delta_{\lambda\mu} \tag{674}$$

will be useful in what follows.

We can use the same set of mode fields to construct both right and left moving waves, except that the relative sign between the longitudinal and transverse fields flips in going from right moving to left moving. Thus if we fix $k_\lambda = +\sqrt{\omega^2\epsilon\mu - \gamma_\lambda^2}$ if real and $k = i\sqrt{\gamma_\lambda^2 - \omega^2\epsilon\mu}$ if imaginary, the right (+) moving and left (−) moving modes will be *defined* via

$$\begin{aligned} \boldsymbol{E}_\lambda^+ &\equiv [\boldsymbol{E}_{\lambda\perp} + \hat{z}E_{\lambda z}]e^{ikz}, \qquad \boldsymbol{H}_\lambda^+ \equiv [\boldsymbol{H}_{\lambda\perp} + \hat{z}H_{\lambda z}]e^{ikz} && (675)\\ \boldsymbol{E}_\lambda^- &\equiv [\boldsymbol{E}_{\lambda\perp} - \hat{z}E_{\lambda z}]e^{-ikz}, \qquad \boldsymbol{H}_\lambda^- \equiv [-\boldsymbol{H}_{\lambda\perp} + \hat{z}H_{\lambda z}]e^{-ikz} && (676)\end{aligned}$$

This notation subsumes both TE and TM modes, the index λ distinguishing different modes. If λ is a TE (TM) mode then $E_{\lambda z} = 0$ ($H_{\lambda z} = 0$). Each mode is associated with its own cutoff frequency $\gamma_\lambda/\sqrt{\mu\epsilon}$.

Returning to the waveguide excitation problem, we can expand the solution of Maxwell's equations outside the source in a compete set of waveguide modes

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}^+ + \boldsymbol{E}^- = \sum_\lambda A_\lambda^+ \boldsymbol{E}_\lambda^+ + \sum_\lambda A_\lambda^- \boldsymbol{E}_\lambda^- && (677)\\ \boldsymbol{H} &= \boldsymbol{H}^+ + \boldsymbol{H}^- = \sum_\lambda A_\lambda^+ \boldsymbol{H}_\lambda^+ + \sum_\lambda A_\lambda^- \boldsymbol{H}_\lambda^- && (678)\end{aligned}$$

Maxwell's equations imply that the expansion coefficients will be a fixed set of constants throughout any region of z completely free of sources. However, if two such regions are separated by a region containing sources, the expansion coefficients will differ for the two regions. Since we are assuming a localized source we have one set of expansion coefficients $A_{R\lambda}^\pm$ in the region to the right of the source and another set $A_{L\lambda}^\pm$ to the left of the source. For the analysis of the waveguide excitation, we shall impose that $\boldsymbol{E}^+ = 0$ ($A_{L\lambda}^+ = 0$ for all λ) everywhere to the left of the source and $\boldsymbol{E}^- = 0$ ($A_{R\lambda}^- = 0$ for all λ) everywhere to the right of the source. The problem is to relate the nonzero coefficients, $A_{L\lambda}^-, A_{R\lambda}^+$ to the properties of the source.

We can then use a trick with the Poynting vector to relate the expansion coefficients on the left to those on the right and a volume integral of the expressions $\boldsymbol{J}\cdot\boldsymbol{E}_\lambda^\pm$. It is based on the identity

$$\begin{aligned} \nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}_\lambda^\pm) &= (\nabla\times\boldsymbol{E})\cdot\boldsymbol{H}_\lambda^\pm - \boldsymbol{E}\cdot(\nabla\times\boldsymbol{H}_\lambda^\pm)\\ &= i\omega\mu\boldsymbol{H}\cdot\boldsymbol{H}_\lambda^\pm + i\omega\epsilon\boldsymbol{E}\cdot\boldsymbol{E}_\lambda^\pm\\ \nabla\cdot(\boldsymbol{E}_\lambda^\pm\times\boldsymbol{H}) &= (\nabla\times\boldsymbol{E}_\lambda^\pm)\cdot\boldsymbol{H} - \boldsymbol{E}_\lambda^\pm\cdot(\nabla\times\boldsymbol{H})\\ &= i\omega\mu\boldsymbol{H}\cdot\boldsymbol{H}_\lambda^\pm + i\omega\epsilon\boldsymbol{E}\cdot\boldsymbol{E}_\lambda^\pm - \boldsymbol{J}\cdot\boldsymbol{E}_\lambda^\pm\\ \nabla\cdot\big(\boldsymbol{E}\times\boldsymbol{H}_\lambda^\pm - \boldsymbol{E}_\lambda^\pm\times\boldsymbol{H}\big) &= \boldsymbol{J}\cdot\boldsymbol{E}_\lambda^\pm && (679)\end{aligned}$$

We can now integrate this over the volume within the waveguide between $-L/2 < z < L/2$ which completely contains the source. The left side can be written as a surface integral over the boundary of the volume. On the walls of the guide the normal component of $\boldsymbol{E} \times \boldsymbol{H}$ involves only the parallel component of $\boldsymbol{E}$ which vanishes on the walls. Thus the only contribution comes from the cross sections at $z = \pm L/2$. At $z = L/2$, $\boldsymbol{E}^- = \boldsymbol{H}^- = 0$, so that contribution is

$$\begin{aligned}
&\sum_\mu A^+_{R\mu} \int d^2x \hat{z} \cdot \left(\boldsymbol{E}^+_\mu \times \boldsymbol{H}^\pm_\lambda - \boldsymbol{E}^\pm_\lambda \times \boldsymbol{H}^+_{R\mu}\right) \\
&\qquad = \sum_\mu A^+_{R\mu} \int d^2x \hat{z} \cdot (\boldsymbol{E}_{\perp\mu} \times (\pm)\boldsymbol{H}_{\perp\lambda} - \boldsymbol{E}_{\perp\lambda} \times \boldsymbol{H}_{\perp\mu}) \\
&\qquad = \sum_\mu A^+_{R\mu} \left(\pm\frac{1}{Z_\lambda}\delta_{\lambda\mu} - \frac{1}{Z_\lambda}\delta_{\lambda\mu}\right) = \begin{cases} 0 \\ -\frac{2}{Z_\lambda} A^+_{R\lambda} \end{cases}
\end{aligned} \tag{680}$$

Similarly at $z = -L/2$, $\boldsymbol{E}^+ = \boldsymbol{H}^+ = 0$, so that contribution is

$$\begin{aligned}
&\sum_\mu A^-_{L\mu} \int d^2x(-\hat{z}) \cdot \left(\boldsymbol{E}^-_\mu \times \boldsymbol{H}^\pm_\lambda - \boldsymbol{E}^\pm_\lambda \times \boldsymbol{H}^-_{\mu}\right) \\
&\qquad = -\sum_\mu A^-_{L\mu} \int d^2x \hat{z} \cdot (\boldsymbol{E}_{\perp\mu} \times (\pm)\boldsymbol{H}_{\perp\lambda} + \boldsymbol{E}_{\perp\lambda} \times \boldsymbol{H}_{\perp\mu}) \\
&\qquad = -\sum_\mu A^-_{L\mu} \left(\pm\frac{1}{Z_\lambda}\delta_{\lambda\mu} + \frac{1}{Z_\lambda}\delta_{\lambda\mu}\right) = \begin{cases} -\frac{2}{Z_\lambda} A^-_{L\lambda} \\ 0 \end{cases}
\end{aligned} \tag{681}$$

Putting the two contributions together gives

$$A^\pm_{\substack{R\\L}\lambda} = -\frac{Z_\lambda}{2}\int d^3x \boldsymbol{J} \cdot \boldsymbol{E}^\mp_\lambda = -\frac{Z_\lambda}{2}\int d^3x \left(\boldsymbol{J}_\perp \cdot \boldsymbol{E}_{\perp\lambda} \pm J_z E_{\lambda z}\right) e^{\mp i k_\lambda z} \tag{682}$$

Thus the strength of the excitation of each mode, propagating or not is determined by this single integral[8]. Far down the wave guide only the propagating modes with $\gamma_\lambda^2 < \omega^2\epsilon\mu$ will survive. For a fixed frequency, there will be only a finite number of these.

[8]More generally if we had not set $A^\pm_{\substack{L\\R}\lambda} = 0$, the left side of these equations would be the differences $A^\pm_{\substack{R\\L}\lambda} - A^\pm_{\substack{L\\R}\lambda}$. So, for instance, if $\boldsymbol{J} = 0$, the equation would simply say that the coefficients on the left and right would be the same: $A^\pm_{R\lambda} = A^\pm_{L\lambda}$.

11 Radiation from Localized Sources

We shall use Green function methods to calculate how electromagnetic waves are emitted from time varying sources. Last semester we obtained the Green function for the wave equation

$$\left(-\nabla^2+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\boldsymbol{r},t;\boldsymbol{r}',t') = \delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t') \tag{683}$$

$$G(\boldsymbol{r},t;\boldsymbol{r}',t') = \frac{\delta(|\boldsymbol{r}-\boldsymbol{r}'|/c-t+t')}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \tag{684}$$

where a choice has been made so that G is zero for $t<t'$, that is: G is the retarded Green function. Maxwell's equations can be reduced to the wave equation by choosing Lorenz gauge $\partial_\mu A^\mu=0$. This is easily seen in covariant notation:

$$J^\mu = c\epsilon_0\partial_\nu F^{\mu\nu}=c^2\epsilon_0\partial_\nu(\partial^\mu A^\nu-\partial^\nu A^\mu)=-\frac{1}{\mu_0}\partial\cdot\partial A^\mu \tag{685}$$

$$-\partial\cdot\partial A^\mu = \left(-\nabla^2+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A^\mu=\mu_0 J^\mu \tag{686}$$

$$A_\mu(\boldsymbol{r},t) = \mu_0\int d^3r'\frac{J_\mu(\boldsymbol{r}',t-R/c)}{4\pi R}, \qquad R\equiv|\boldsymbol{r}-\boldsymbol{r}'| \tag{687}$$

where we note that the source contributes at the retarded time.

Radiation, by definition, is electromagnetic energy that escapes to large distances, and is determined by the large r behavior of the integral. We can write $R\approx r-\boldsymbol{r}\cdot\boldsymbol{r}'/r$

$$A_\mu(\boldsymbol{r},t) \sim \frac{\mu_0}{4\pi r}\int d^3r' J_\mu(\boldsymbol{r}',t-R/c), \qquad r\gg a \tag{688}$$

If we Fourier analyze $\boldsymbol{J}(\boldsymbol{r},t)$ in time we get for the component with frequency ω,

$$\begin{aligned} A_\mu(\boldsymbol{r},\omega)e^{-i\omega t} &\sim e^{-i\omega t}\frac{\mu_0}{4\pi r}\int d^3r' J_\mu(\boldsymbol{r}',\omega)e^{ik|\boldsymbol{r}-\boldsymbol{r}'|} \\ A_\mu(\boldsymbol{r},\omega) &\sim \frac{\mu_0 e^{ikr}}{4\pi r}\int d^3r' J_\mu(\boldsymbol{r}',\omega)e^{-ik\boldsymbol{r}\cdot\boldsymbol{r}'/r} \end{aligned} \tag{689}$$

The simplest radiating physical systems are point particles with harmonic trajectories, for instance moving in a circle of radius a in the xy-plane at constant speed ωa, $\boldsymbol{r}(t)=a(\cos(\omega t),\sin(\omega t,0)$. The current for such a particle is

$$\boldsymbol{J} = q\frac{d\boldsymbol{r}}{dt}\delta(\boldsymbol{r}-\boldsymbol{r}(t)) \tag{690}$$

$$\begin{aligned} J_x\pm iJ_y &= \pm iqa\omega e^{\pm i\omega t}\delta(r-a)\delta(z)\frac{1}{a}\delta(\varphi-\omega t) \\ &= \pm iq\omega e^{\pm i\omega t}\delta(r-a)\delta(z)\delta(\varphi-\omega t), \qquad J_z=0 \end{aligned} \tag{691}$$

The angle delta function in this formula must be understood as requiring $\varphi = \omega t \bmod 2\pi$, which means it has the Fourier representation

$$\delta(\varphi - \omega t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in(\varphi-\omega t)} \tag{692}$$

Thus we see that although the motion has the frequency ω, the current oscillates at all the harmonics $n\omega$ of this fundamental frequency.

$$J_x \pm iJ_y = \pm\frac{iq\omega}{2\pi}e^{\pm i\omega t}\delta(r-a)\delta(z)\sum_{n=-\infty}^{\infty} e^{in(\varphi-\omega t-\varphi_0)} \tag{693}$$

But notice that the higher harmonics are accompanied with high powers of $e^{i\varphi}$, which means that they contribute only to higher multipole moments. In the homework problems you are asked to analyze dipole and quadrupole arrangements of such revolving charges.

To calculate the Poynting vector which describes the energy flow of radiation, we need the fields.

$$\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} = \frac{1}{\mu_0}\nabla\times\boldsymbol{A} \tag{694}$$

$$\begin{aligned}\boldsymbol{E} &= -\frac{\partial\boldsymbol{A}}{\partial t} - \nabla\phi = i\omega\boldsymbol{A} - \nabla\phi \\ &= i\omega\boldsymbol{A} - \frac{c^2}{i\omega}\nabla(\nabla\cdot\boldsymbol{A})\end{aligned} \tag{695}$$

where the last line uses the Lorenz gauge condition $0 = \partial A^0/\partial(ct) + \nabla\cdot\boldsymbol{A} \to -i\omega\phi/c^2 + \nabla\cdot\boldsymbol{A}$. In the radiation zone, characterized by the fields falling off as $1/r$, the only contribution to the spatial derivatives which does not add a further factor of $1/r$ is when they act on the phase factor $\nabla e^{ikr} = ik\boldsymbol{r}e^{ikr}/r \equiv i\boldsymbol{k}e^{ikr}$. Thus, in the radiation zone characterized by $kr \gg 1$, it is valid to substitute $\nabla \to i\boldsymbol{k}$.

$$\boldsymbol{H} \sim i\frac{e^{ikr}}{4\pi r}\boldsymbol{k}\times\int d^3r'\boldsymbol{J}(\boldsymbol{r}',\omega)e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'} \tag{696}$$

$$\begin{aligned}\boldsymbol{E} &\sim \frac{\mu_0 e^{ikr}}{4\pi\omega r}\int d^3r'\left(i\omega^2\boldsymbol{J}(\boldsymbol{r}',\omega) - ic^2\boldsymbol{k}\boldsymbol{k}\cdot\boldsymbol{J}(\boldsymbol{r}',\omega)\right)e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'} \\ &\sim -i\frac{e^{ikr}}{4\pi\epsilon_0\omega r}\boldsymbol{k}\times\left(\boldsymbol{k}\times\int d^3r'\boldsymbol{J}(\boldsymbol{r}',\omega)e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'}\right) = -Z_0\hat{\boldsymbol{k}}\times\boldsymbol{H}\end{aligned} \tag{697}$$

where $Z_0 = \mu_0 c = \sqrt{\mu_0/\epsilon_0} \approx 377\Omega$ is sometimes called the impedance of the vacuum. Note that, had we not used the Lorenz gauge condition to relate ϕ to $\boldsymbol{A}$, the integrand would involve

$$\begin{aligned}i\omega\boldsymbol{J}(\boldsymbol{r}',\omega) - i\boldsymbol{k}c^2\rho(\boldsymbol{r}',\omega) &= i\omega\boldsymbol{J}(\boldsymbol{r}',\omega) - \frac{\boldsymbol{k}c^2}{\omega}\nabla'\cdot\boldsymbol{J}(\boldsymbol{r}',\omega) \\ &\to i\omega\boldsymbol{J} - i\frac{\boldsymbol{k}c^2}{\omega}\boldsymbol{k}\cdot\boldsymbol{J} = -i\frac{c^2}{\omega}\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{J})\end{aligned} \tag{698}$$

where we used current conservation $\nabla \cdot \boldsymbol{J} = i\omega\rho$ in the first line and the arrow denotes the change in the integrand after an integration by parts. Finally the flux of energy in the radiation is given by the time averaged Poynting vector, which at large r is

$$\begin{aligned}
\langle \boldsymbol{S} \rangle &= \frac{1}{2}\text{Re } \boldsymbol{E} \times \boldsymbol{H}^* \\
&\sim -\frac{Z_0}{2k}\text{Re } (\boldsymbol{k} \times \boldsymbol{H}) \times \boldsymbol{H}^* = \frac{Z_0 \boldsymbol{k}}{2k} \boldsymbol{H} \cdot \boldsymbol{H}^* \\
\frac{dP}{d\Omega} &= r^2 \hat{r} \cdot \langle \boldsymbol{S} \rangle \sim \frac{Z_0}{32\pi^2} \left| \boldsymbol{k} \times \int d^3r' \boldsymbol{J}(\boldsymbol{r'}, \omega) e^{-i\boldsymbol{k}\cdot\boldsymbol{r'}} \right|^2 \\
&= \frac{Z_0}{32\pi^2 k^2} \left| \boldsymbol{k} \times \left(\boldsymbol{k} \times \int d^3r' \boldsymbol{J}(\boldsymbol{r'}, \omega) e^{-i\boldsymbol{k}\cdot\boldsymbol{r'}} \right) \right|^2
\end{aligned} \tag{699}$$

The main point of using the formula on the last line is that the vector inside the absolute value points in the direction of the electric field and hence determines the polarization of the radiation.

11.1 Long Wavelength Limit

We have just seen that the intensity of radiation involves the Fourier transform of the current density

$$\int d^3r \boldsymbol{J}(\boldsymbol{r}, \omega) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{700}$$

When only one frequency contributes the normalization is that $\text{Re}(\boldsymbol{J}e^{-i\omega t})$ is the actual physical current density. Suppose the source is localized within a region of size a. Then the exponent is of order $ka = 2\pi a/\lambda$. If the wavelength is large compared to the size of the source, then $ka \ll 1$ and we may expand the exponential, relating the terms of the expansion to various multipole moments. We examine the first few terms of this expansion.

$$e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} = 1 - i\boldsymbol{k} \cdot \boldsymbol{r} - \frac{1}{2}(\boldsymbol{k} \cdot \boldsymbol{r})^2 + \cdots \tag{701}$$

The first term contributes

$$\int d^3r J^k(\boldsymbol{r}) = \int d^3r \frac{\partial r^k}{\partial r^l} J^l(\boldsymbol{r}) = -\int d^3r r^k \nabla \cdot \boldsymbol{J}(\boldsymbol{r}) = -i\omega \int d^3r r^k \rho = -i\omega p^k \tag{702}$$

where $\boldsymbol{p}$ is the electric dipole moment.

The next term contributes

$$\begin{aligned}
-ik_l \int d^3r r^l J^k &= -i\frac{k^l}{2} \int d^3r (r^l J^k + r^k J^l) - \frac{i}{2} \int d^3r (\boldsymbol{r} \cdot \boldsymbol{k} J^k - r^k \boldsymbol{k} \cdot \boldsymbol{J}) \\
&= -i\frac{k^l}{2} \int d^3r J^m \nabla_m (r^k r^l) - \frac{i}{2} \int d^3r [\boldsymbol{k} \times (\boldsymbol{J} \times \boldsymbol{r})]^k
\end{aligned}$$

$$
\begin{aligned}
&= i\frac{k^l}{2}\int d^3r r^k r^l \nabla\cdot\boldsymbol{J} + i\boldsymbol{k}\times\boldsymbol{m} = -\omega\frac{k^l}{2}\int d^3r r^k r^l\rho + i\boldsymbol{k}\times\boldsymbol{m} \\
&= -\omega\frac{k^l}{2}\left(\frac{Q_{kl}}{3}+\frac{\delta_{kl}}{3}\int d^3r r^2\rho\right) + i[\boldsymbol{k}\times\boldsymbol{m}]^k \qquad (703)
\end{aligned}
$$

where we have recalled the definition of the electric quadrupole moment $Q_{kl} = \int d^3x(3r^k r^l - r^2\delta_{kl})\rho$. Finally, we have

$$
\int d^3r J^k(\boldsymbol{r})e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} = -i\omega p^k + i[\boldsymbol{k}\times\boldsymbol{m}]^k - \frac{\omega}{6}Q_{kl}k^l - \frac{\omega k^k}{6}\int d^3r r^2\rho \qquad (704)
$$

The last term, which is not expressed in terms of a multipole moment does not contribute to radiation which involves the vector product of the current with $\boldsymbol{k}$. The three terms that do contribute are electric dipole, magnetic dipole, and electric quadrupole respectively. Each type of radiation has a characteristic angular distribution.

For electric and magnetic dipole radiation one has

$$
\begin{aligned}
\frac{\omega^2}{k^2}|\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{p})|^2 &= \frac{\omega^2}{k^2}|\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{p}) - k^2\boldsymbol{p}|^2 = \omega^2(k^2\boldsymbol{p}\cdot\boldsymbol{p}^* - \boldsymbol{k}\cdot\boldsymbol{p}\boldsymbol{k}\cdot\boldsymbol{p}^*) \\
\frac{1}{k^2}|\boldsymbol{k}\times(\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{m}))|^2 &= k^2(k^2\boldsymbol{m}\cdot\boldsymbol{m}^* - \boldsymbol{k}\cdot\boldsymbol{m}\boldsymbol{k}\cdot\boldsymbol{m}^*) \qquad (705)
\end{aligned}
$$

Only if the components of $\boldsymbol{p}$ or $\boldsymbol{m}$ are all relatively real can we identify the right sides of these equations with $\omega^2k^2|\boldsymbol{p}|^2\sin^2\alpha$ or $k^4|\boldsymbol{m}|^2\sin^2\alpha$, where α is the angle between $\boldsymbol{k}$ and the dipole moment.

$$
\frac{dP}{d\Omega} = \frac{Z_0}{32\pi^2}\omega^2(k^2|\boldsymbol{p}|^2 - |\boldsymbol{k}\cdot\boldsymbol{p}|^2), \qquad \frac{dP}{d\Omega} = \frac{Z_0}{32\pi^2}k^2(k^2|\boldsymbol{m}|^2 - |\boldsymbol{k}\cdot\boldsymbol{m}|^2) \qquad (706)
$$

The only qualitative difference between electric and magnetic dipole radiation is the polarization, i.e. the orientation of the electric field. For the electric dipole it points in the direction $\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{p})$ which lies in the plane defined by $\boldsymbol{k}$ and $\boldsymbol{p}$. For the magnetic dipole it points in the direction perpendicular to the plane defined by $\boldsymbol{k}$ and $\boldsymbol{m}$. The total power radiated requires $\int d\Omega k^i k^j = 4\pi k^2/3\delta_{ij}$

$$
P = \frac{Z_0}{12\pi}\omega^2k^2|\boldsymbol{p}|^2 = \frac{Z_0}{12\pi}k^4c^2|\boldsymbol{p}|^2, \qquad P = \frac{Z_0}{12\pi}k^4|\boldsymbol{m}|^2 \qquad (707)
$$

respectively.

Quadrupole radiation has a more complicated angular distribution. We have to square the vector with components $V^i = \epsilon_{ijk}k^jQ_{kl}k^l$

$$
\begin{aligned}
|\boldsymbol{V}|^2 &= \epsilon_{ijk}k^jQ_{kl}k^l\epsilon_{imn}k^mQ^*_{np}k^p = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})k^jQ_{kl}k^lk^mQ^*_{np}k^p \\
&= \boldsymbol{k}^2Q_{kl}Q^*_{kp}k^lk^p - |Q_{ml}k^mk^l|^2 \qquad (708) \\
\frac{dP}{d\Omega} &= \frac{\omega^2k^4Z_0}{1152\pi^2}(Q_{ml}Q^*_{mp}\hat{k}^l\hat{k}^p - |Q_{ml}\hat{k}^m\hat{k}^l|^2) \qquad (709)
\end{aligned}
$$

where $\hat{k} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$.

To see the angular distribution in a simple example, take Q_{ij} diagonal with $Q_{33} = -2Q_{11} = -2Q_{22}$. Then $Q_{ij}Q_{in}\hat{k}^j\hat{k}^n/Q_{11}^2 = \sin^2\theta + 4\cos^2\theta = 1 + 3\cos^2\theta$ and $Q_{ij}\hat{k}^i\hat{k}^j/Q_{11} = \sin^2\theta - 2\cos^2\theta = 1 - 3\cos^2\theta$. then

$$\frac{dP}{d\Omega} = 9\frac{c^2k^6Z_0}{1152\pi^2}Q_{11}^2\sin^2\theta\cos^2\theta = \frac{c^2k^6Z_0}{128\pi^2}Q_{11}^2\sin^2\theta\cos^2\theta. \quad (710)$$

This shows a four lobed structure with vanishing radiation at $\theta = 0, \pi/2$.

To integrate over angles one can use rotational invariance and symmetry to argue that

$$\int d\Omega k^m k^l = C_1\delta_{ml}, \qquad \int d\Omega k^m k^l k^n k^p = C_2(\delta_{ml}\delta_{np} + \delta_{mn}\delta_{lp} + \delta_{mp}\delta_{ln}) \quad (711)$$

Then tracing over ml determines $C_1 = 4\pi/3$, $5C_2 = C_1$ so $C_2 = 4\pi/15$. Then we find

$$P = \frac{4\omega^2k^4Z_0}{1152\pi}\left(\frac{1}{3}\sum_{ml}|Q_{ml}|^2 - \frac{1}{15}(|\mathrm{Tr}Q|^2 + 2\sum_{ml}|Q_{ml}|^2)\right) = \frac{\omega^2k^4Z_0}{1440\pi}\sum_{ml}|Q_{ml}|^2 \quad (712)$$

for the total readiated power.

11.2 Beyond the Multipole Expansion

If the wavelength of radiation is comparable to the size of the source, the multipole expansion is no longer valid, and we have to try to evaluate the full Fourier transform $\int d^3x\boldsymbol{J}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}$. For a line antenna we can for example prescribe a simple form for $\boldsymbol{J} = \hat{z}I\delta(x)\delta(y)\sin(kd/2-k|z|)$ for $-d/2 < z < d/2$. Then

$$\begin{aligned}
&\int d^3x\boldsymbol{J}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} = \hat{z}I\int_{-d/2}^{d/2}dz\sin\left(\frac{kd}{2} - k|z|\right)e^{-ikz\cos\theta} \\
&= \frac{I\hat{z}}{2i}\int_0^{d/2}dz\left(e^{ikd/2-ikz(1-\cos\theta)} + e^{ikd/2-ikz(1+\cos\theta)} - e^{-ikd/2+ikz(1+\cos\theta)} - e^{-ikd/2+ikz(1-\cos\theta)}\right) \\
&= \frac{I\hat{z}}{2i}\left(\frac{e^{i(kd/2)\cos\theta)} + e^{-i(kd/2)\cos\theta)} - e^{ikd/2} - e^{-ikd/2}}{-ik(1-\cos\theta)} + \frac{e^{i(kd/2)\cos\theta)} + e^{-i(kd/2)\cos\theta)} - e^{ikd/2} - e^{-ikd/2}}{-ik(1+\cos\theta)}\right) \\
&= \frac{2I\hat{z}}{k}\frac{\cos[(kd/2)\cos\theta] - \cos(kd/2)}{1-\cos^2\theta} = \frac{2I\hat{z}}{k}\frac{\cos[(kd/2)\cos\theta] - \cos(kd/2)}{\sin^2\theta} \quad (713)
\end{aligned}$$

The angular power distribution is

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{Z_0}{32\pi^2}\frac{4I^2}{k^4}\left|\frac{\cos[(kd/2)\cos\theta] - \cos(kd/2)}{\sin^2\theta}\right|^2|\boldsymbol{k}\times(\boldsymbol{k}\times\hat{z})|^2 \\
&= \frac{Z_0I^2}{8\pi^2}\left|\frac{\cos[(kd/2)\cos\theta] - \cos(kd/2)}{\sin\theta}\right|^2 \quad (714)
\end{aligned}$$

If $kd \ll 1$ we can expand in powers of kd which generates a multipole expansion. The numerator

$$\begin{aligned}
\cos[(kd/2)\cos\theta] - \cos(kd/2) &= \sum_{n=0}^{\infty} \frac{(-)^n}{(2n)!}(kd/2)^{2n}(\cos^{2n}\theta - 1) \\
&= -\sin^2\theta \sum_{n=1}^{\infty} \frac{(-)^n}{(2n)!}(kd/2)^{2n} \sum_{m=0}^{n-1} \cos^{2m}\theta \\
&\sim \frac{k^2d^2}{8}\sin^2\theta - \frac{k^4d^4}{384}\sin^2\theta(1+\cos^2\theta) + \cdots \qquad (715)
\end{aligned}$$

We see that the multipole expansion of the angular distribution will be $\sin^2\theta$ times a power series in $\cos^2\theta$.

But the formula also gives us a closed form expression even when $kd = O(1)$. For example if $kd = \pi, 2\pi$ the formula simplifies to

$$\frac{dP}{d\Omega}_{kd=\pi} = \frac{Z_0 I^2}{8\pi^2}\frac{\cos^2[(\pi/2)\cos\theta]}{\sin^2\theta}, \qquad \frac{dP}{d\Omega}_{kd=2\pi} = \frac{Z_0 I^2}{8\pi^2}\frac{4\cos^4[(\pi/2)\cos\theta]}{\sin^2\theta} \qquad (716)$$

The intensity at $\theta = \pi/2$ is 4 times larger in the 2π case than in the π case. On the other hand as $\theta \to 0, \pi$ the intensity drops to zero much faster in the 2π case.

11.3 Systematics of the Multipole Expansion

The way to systematically handle multipole radiation is to expand the angular dependence in spherical harmonics. The Helmholtz equation satisfied by all 6 field components separates in spherical coordinates in the same way as the Laplace equation. We seek solutions of the form $R(r)Y_{lm}(\theta, \varphi)$ and find the radial equation

$$\left(-\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{l(l+1)}{r^2} - k^2\right) R = 0 \tag{717}$$

This equation can be compared to the Bessel equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + k^2 - \frac{m^2}{r^2}\right) J_m(kr) = \left(\frac{1}{\sqrt{r}}\frac{d^2}{dr^2}\sqrt{r} + k^2 - \frac{m^2 - 1/4}{r^2}\right) J_m(kr) = 0 \tag{718}$$

We see that the solution is $R = \frac{1}{\sqrt{r}} J_{l+1/2}(kr)$. These special Bessel functions are called spherical Bessel functions and are denoted by lower case letters:

$$\begin{aligned} j_l(x) &\equiv \sqrt{\frac{\pi}{2x}} J_{l+1/2}(x) \sim \frac{\cos(x - (l+1)\pi/2)}{x}, \\ n_l(x) &\equiv \sqrt{\frac{\pi}{2x}} N_{l+1/2}(x) \sim \frac{\sin(x - (l+1)\pi/2)}{x}, \\ h_l^{(1,2)} &\equiv j_l \pm i n_l \sim \frac{\exp(\pm i(x-(l+1)\pi/2))}{x} \end{aligned} \tag{719}$$

where the asymptotics shown is for $x >> 1$. Amazingly the spherical Bessel functions are rational functions of x and linear functions of $\sin x, \cos x$. E.g.

$$j_0(x) = \frac{\sin x}{x}, \qquad n_0(x) = -\frac{\cos x}{x}, \qquad h_0^{(1)}(x) = -i\frac{e^{ix}}{x} \tag{720}$$

By a familiar argument we have the spherical expansion of the Green function for the Helmholtz equation:

$$\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} = ik\sum_{l=0}^{\infty} h_l^{(1)}(kr_>) j_l(kr_<) \sum_{m=-l}^{l} Y_{lm}(\theta,\varphi) Y_{lm}^*(\theta',\varphi') \tag{721}$$

This expansion immediately gives an expansion in spherical waves for the potential everywhere completely outside a harmonically varying source so $r > r'$:

$$\begin{aligned} A_\mu(\boldsymbol{r},\omega) &= \mu_0 \int d^3r' J_\mu(\boldsymbol{r}') \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \\ &= ik\mu_0 \sum_{l=0}^{\infty}\sum_{m=-l}^{l} h_l^{(1)}(kr) Y_{lm}(\theta,\varphi) \int d^3r' J_\mu(\boldsymbol{r}') j_l(kr') Y_{lm}^*(\theta',\varphi') && (722) \\ &\sim \frac{e^{ikr}}{r}\mu_0 \sum_{l=0}^{\infty}(-i)^l \sum_{m=-l}^{l} Y_{lm}(\theta,\varphi) \int d^3r' J_\mu(\boldsymbol{r}') j_l(kr') Y_{lm}^*(\theta',\varphi') && (723) \end{aligned}$$

in the radiation zone. Notice that the strength of each Y_{lm} is not literlly a multipole moment because the r^l of the latter has been replaced by $j_l(kr)$. But they become multipole moments s $kr \to 0$, the low frequency limit. This expansion does not quite a completely organize the angular dependence of the fields, because of the angular information contained in $\boldsymbol{J}$ and in the derivatives relating the potential to the fields. Next we turn to a more elegant description.

11.4 Vector Spherical Harmonics and Multipole Radiation

We can construct a vector from the components of Y_{lm} by applying the angular momentum operator $\boldsymbol{L} = -i\boldsymbol{r} \times \nabla$. This suggests introducing vector spherical harmonics $\boldsymbol{X}_{lm} \equiv \boldsymbol{L}Y_{lm}/\sqrt{l(l+1)}$. Since $\boldsymbol{L}$ does not change the l value, the components of $\boldsymbol{X}_{lm}$ are linear combinations of the $Y_{lm'}$, so $f_l(kr)\boldsymbol{X}_{lm}$ satisfies the Helmholtz equation if f_l is a spherical Bessel function. Furthermore $\nabla \cdot \boldsymbol{X}_{lm} = 0$ and $\boldsymbol{r} \cdot \boldsymbol{X}_{lm} = 0$, which together imply that $\nabla \cdot (f_l(kr)\boldsymbol{X}_{lm}) = 0$. Thus $f_l(kr)\boldsymbol{X}_{lm}$ is a candidate for either $\boldsymbol{E}$ or $\boldsymbol{H}$ outside of sources. Using $ik\boldsymbol{H} = \sqrt{\epsilon/\mu}\nabla \times \boldsymbol{E}$ or $-ik\boldsymbol{E} = \sqrt{\mu/\epsilon}\nabla \times \boldsymbol{H}$ then determines the expansions for a general solution of the sourceless Maxwell equations:

$$\boldsymbol{H} = \sum_{lm}\left[a^E_{lm} f_l(kr)\boldsymbol{X}_{lm} - \frac{i}{k}a^M_{lm}\nabla \times (g_l(kr)\boldsymbol{X}_{lm})\right] \tag{724}$$

$$\boldsymbol{E} = Z_0\sum_{lm}\left[\frac{i}{k}a^E_{lm}\nabla \times f_l(kr)\boldsymbol{X}_{lm} + a^M_{lm}g_l(kr)\boldsymbol{X}_{lm})\right] \tag{725}$$

If all $a^M = 0$, the fields satisfy $\boldsymbol{r} \cdot \boldsymbol{H} = 0$ and $\boldsymbol{r} \cdot \boldsymbol{E} \neq 0$, so they have the quality of *electric* multipoles. With all $a^E = 0$, the properties of $\boldsymbol{E}$, $\boldsymbol{H}$ are switched, and the fields have the quality of *magnetic* multipoles.

To determine a^E, a^M in terms of sources, note that

$$\boldsymbol{r} \cdot \boldsymbol{H} = -\frac{i}{k}\sum_{lm} a^M_{lm} i\boldsymbol{L} \cdot (g_l(kr)\boldsymbol{X}_{lm}) = \frac{1}{k}\sum_{lm} a^M_{lm}\sqrt{l(l+1)}g_l(kr)Y_{lm} \tag{726}$$

$$\boldsymbol{r} \cdot \boldsymbol{E} = -\frac{Z_0}{k}\sum_{lm} a^E_{lm}\sqrt{l(l+1}f_l(kr)Y_{lm} \tag{727}$$

$$g_l(kr)a^M_{lm} = \frac{k}{\sqrt{l(l+1)}}\int d\Omega Y^*_{lm}\boldsymbol{r} \cdot \boldsymbol{H},$$

$$Z_0 f_l(kr)a^E_{lm} = \frac{-k}{\sqrt{l(l+1)}}\int d\Omega Y^*_{lm}\boldsymbol{r} \cdot \boldsymbol{E} \tag{728}$$

The Helmholtz equations for the fields

$$(-\nabla^2 - k^2)\boldsymbol{H} = \nabla \times \boldsymbol{J}$$

$$(-\nabla^2 - k^2)\boldsymbol{E} = \frac{iZ_0}{k}(k^2\boldsymbol{J} + \nabla\nabla \cdot \boldsymbol{J}), \tag{729}$$

where the second equation follows by taking the curl of the first one. From these, we can derive Helmholtz equations for $\boldsymbol{r}\cdot\boldsymbol{H}, \boldsymbol{r}\cdot\boldsymbol{E}$:

$$\begin{aligned}
(-\nabla^2-k^2)\boldsymbol{r}\cdot\boldsymbol{H} &= \boldsymbol{r}\cdot(-\nabla^2-k^2)\boldsymbol{H}-2\nabla\cdot\boldsymbol{H}\\
&= \boldsymbol{r}\cdot(\nabla\times\boldsymbol{J}) = i\boldsymbol{L}\cdot\boldsymbol{J} \qquad (730)\\
(-\nabla^2-k^2)\boldsymbol{r}\cdot\boldsymbol{E} &= \boldsymbol{r}\cdot(-\nabla^2-k^2)\boldsymbol{E}-2\nabla\cdot\boldsymbol{E}\\
&= \frac{iZ_0}{k}\boldsymbol{r}\cdot(k^2\boldsymbol{J}+\nabla\nabla\cdot\boldsymbol{J})-2\frac{Z_0}{ik}\nabla\cdot\boldsymbol{J}\\
&= \frac{iZ_0}{k}\boldsymbol{r}\cdot[(k^2+\nabla^2)\boldsymbol{J}+\nabla\times(\nabla\times\boldsymbol{J})]+2\frac{iZ_0}{k}\nabla\cdot\boldsymbol{J}\\
&= \frac{iZ_0}{k}[(k^2+\nabla^2)\boldsymbol{r}\cdot\boldsymbol{J}+i\boldsymbol{L}\cdot(\nabla\times\boldsymbol{J})] \qquad (731)
\end{aligned}$$

Solving these equations with outgoing radiation boundary conditions

$$\begin{aligned}
\boldsymbol{r}\cdot\boldsymbol{H} &= \int d^3r'\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}i\boldsymbol{L}\cdot\boldsymbol{J} \qquad (732)\\
\boldsymbol{r}\cdot\boldsymbol{E} &= -\frac{iZ_0}{k}\boldsymbol{r}\cdot\boldsymbol{J}-\frac{Z_0}{k}\int d^3r'\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{L}\cdot(\nabla\times\boldsymbol{J}) \qquad (733)
\end{aligned}$$

Letting $\boldsymbol{r}$ be outside all sources, so that $r>r'$ and $\boldsymbol{J}(\boldsymbol{r},\omega=0)$, we insert the expansion of the Green function in spherical harmonics to get

$$\begin{aligned}
\boldsymbol{r}\cdot\boldsymbol{H} &= -k\sum_{l=0}^{\infty}\sum_{m=-l}^{l}h_l^{(1)}(kr)Y_{lm}(\theta,\varphi)\int d^3r'j_l(kr')Y^*_{lm}(\theta',\varphi')\boldsymbol{L}\cdot\boldsymbol{J} \qquad (734)\\
\boldsymbol{r}\cdot\boldsymbol{E} &= -iZ_0\sum_{l=0}^{\infty}\sum_{m=-l}^{l}h_l^{(1)}(kr)Y_{lm}(\theta,\varphi)\int d^3r'j_l(kr')Y^*_{lm}(\theta',\varphi')\boldsymbol{L}\cdot(\nabla\times\boldsymbol{J}) \qquad (735)
\end{aligned}$$

from which we read off $g_l=f_l=h_l^{(1)}(kr)$ and

$$\begin{aligned}
a^M_{lm} &= \frac{-k^2}{\sqrt{l(l+1)}}\int d^3r'j_l(kr')Y^*_{lm}\boldsymbol{L}\cdot\boldsymbol{J} = -k^2\int d^3r'j_l\boldsymbol{X}^*_{lm}\cdot\boldsymbol{J} \qquad (736)\\
a^E_{lm} &= \frac{ik}{\sqrt{l(l+1)}}\int d^3r'j_l(kr')Y^*_{lm}\boldsymbol{L}\cdot(\nabla\times\boldsymbol{J})\\
&= ik\int d^3r'j_l\boldsymbol{X}^*_{lm}\cdot(\nabla\times\boldsymbol{J}) \qquad (737)
\end{aligned}$$

Note that at finite k these are not strict lth moments of the current densities. But in the long wavelength limit $j_l(kr')\to(kr')^l/(2l+1)!!$ and they do approach pure lth moments. with these definitions the fields outside the source but produced by the source $\boldsymbol{J}$ are

$$\begin{aligned}
\boldsymbol{H} &= \sum_{lm}\left[a^E_{lm}h_l^{(1)}(kr)\boldsymbol{X}_{lm}-\frac{i}{k}a^M_{lm}\nabla\times(h_l^{(1)}(kr)\boldsymbol{X}_{lm})\right] \qquad (738)\\
\boldsymbol{E} &= Z_0\sum_{lm}\left[\frac{i}{k}a^E_{lm}\nabla\times h_l^{(1)}(kr)\boldsymbol{X}_{lm}+a^M_{lm}h_l^{(1)}(kr)\boldsymbol{X}_{lm})\right] \qquad (739)
\end{aligned}$$

These expressions are exact, except for the stipulation that the observation point is outside the source. Going to the radiation zone $kr \gg 1$ we use $h_l^{(1)}(kr) \to (-i)^{l+1} e^{ikr}/kr$, and the fact that in the leading behavior ∇ acts only on the factor e^{ikr}, $\nabla e^{ikr} = ik\hat{r}e^{ikr} \equiv i\boldsymbol{k}e^{ikr}$, we find

$$\boldsymbol{H} \sim \frac{e^{ikr}}{kr}\sum_{lm}(-i)^{l+1}\left[a^E_{lm}\boldsymbol{X}_{lm} + a^M_{lm}\hat{r}\times\boldsymbol{X}_{lm}\right] \tag{740}$$

$$\boldsymbol{E} \sim Z_0\frac{e^{ikr}}{kr}\sum_{lm}(-i)^{l+1}\left[-a^E_{lm}\hat{r}\times\boldsymbol{X}_{lm} + a^M_{lm}\boldsymbol{X}_{lm}\right] \sim -Z_0\hat{r}\times\boldsymbol{H} \tag{741}$$

which determine $dP/d\Omega$ through the time averaged Poynting vector.

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|\sum_{lm}(-i)^{l+1}\left[a^E_{lm}\boldsymbol{X}_{lm} + a^M_{lm}\hat{r}\times\boldsymbol{X}_{lm}\right]\right|^2 \tag{742}$$

When a single multipole contributes (e.g. in the long wavelength limit when the lowest nonvanishing multipole dominates) the angular distribution is carried by

$$|\boldsymbol{X}_{lm}|^2 = \frac{1}{l(l+1)}\left(\frac{1}{2}|L_+Y_{lm}|^2 + \frac{1}{2}|L_-Y_{lm}|^2 + m^2|Y_{lm}|^2\right) \tag{743}$$

Example: $lm = 11$: $L_+Y_{11} = 0$, $L^-Y_{11} = Y_{10}\sqrt{2}$ so

$$|\boldsymbol{X}_{11}|^2 = \frac{1}{4}2|Y_{10}|^2 + \frac{1}{2}|Y_{11}|^2 = \frac{3}{8\pi}\cos^2\theta + \frac{3}{16\pi}\sin^2\theta = \frac{3}{16\pi}(1+\cos^2\theta)$$

We get the total radiated power by integrating over angles. The vector spherical harmonics satisfy orthonormality conditions

$$\int d\Omega \boldsymbol{X}^*_{l'm'}\cdot\boldsymbol{X}_{lm} = \delta_{l'l}\delta_{m'm} \tag{744}$$

$$\begin{aligned}\int d\Omega \boldsymbol{X}^*_{l'm'}\cdot(\hat{r}\times\boldsymbol{X}_{lm}) &= \frac{1}{\sqrt{l(l+1)l'(l'+1)}}\int d\Omega Y^*_{l'm'}\boldsymbol{L}\cdot(\hat{r}\times\boldsymbol{L})Y_{lm}\\ &= \frac{1}{\sqrt{l(l+1)l'(l'+1)}}\int d\Omega Y^*_{l'm'}\boldsymbol{L}\cdot(-i\boldsymbol{r}\hat{r}\cdot\nabla + ir\nabla)Y_{lm}\\ &= \frac{1}{\sqrt{l(l+1)l'(l'+1)}}\int d\Omega Y^*_{l'm'}\cdot\left(-i\boldsymbol{L}\cdot\boldsymbol{r}\frac{\partial Y_{lm}}{\partial r} + ir\boldsymbol{L}\cdot\nabla Y_{lm}\right)\\ &= 0\end{aligned} \tag{745}$$

because $\boldsymbol{L}\cdot\nabla = -i\boldsymbol{r}\cdot(\nabla\times\nabla) = 0$ and $\partial Y_{lm}/\partial r = 0$. This means that the angular integral over all the cross terms is 0, so we have the simple result:

$$P = \frac{Z_0}{2k^2}\sum_{lm}\left[|a^E_{lm}|^2 + |a^M_{lm}|^2\right] , \tag{746}$$

12 Scattering of Electromagnetic Waves

So far we have described how systems with prescribed current density radiate, but the manner in which such current densities can be set up still needs to be understood. One way of course is to drive current in an antenna, say for the purpose of broadcast. But spectroscopic observation of the radiation from atomic systems is an important tool for studying those systems. To radiate the systems must be excited and a basic way to do that is to scatter photons (or other fundamental particles like electrons or neutrons) off them. Here we consider the scattering of classical electromagnetic waves.

The basic quantity that characterizes the outcome of a scattering process is the differential cross section which is the angular distribution of outgoing radiation normalized to unit flux. For example if the outgoing radiation is measured as power per unit solid angle, the differential cross section is obtained by dividing by the incident energy flux: energy per unit area per unit time incident on the system.

12.1 Long Wavelength Scattering

To start with a very simple example, consider a long wavelength photon scattering from a nonrelativistic particle. The particle responds to the fields in the wave according to

$$-m\omega^2\boldsymbol{r} \;=\; q\boldsymbol{E}e^{-i\omega t} \tag{747}$$

where we assume that the velocities stay small so the magnetic force is negligible and that the amplitude of vibration is small compared with the wavelength so the particle only sees a uniform harmonically varying field. Then the induced dipole moment is $\boldsymbol{p} = -q^2\boldsymbol{E}/m\omega^2$, where we omit the $e^{-i\omega t}$ factor. This oscillating dipole emits dipole radiation according to

$$\frac{dP}{d\Omega} \;=\; \frac{Z_0c^2k^4}{32\pi^2}|\hat{r}\times(\hat{r}\times\boldsymbol{p})|^2 = \frac{q^4Z_0c^2k^4}{32\pi^2m^2\omega^4}|\hat{r}\times(\hat{r}\times\boldsymbol{E})|^2 = \frac{q^4Z_0}{32\pi^2m^2c^2}|\hat{r}\times(\hat{r}\times\boldsymbol{E})|^2$$

The time averaged Poynting vector of the incident wave $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{\varepsilon}_0 Ee^{i\boldsymbol{k}_0\cdot\boldsymbol{r}-i\omega t}$, where we normalize $\boldsymbol{\varepsilon}_0\cdot\boldsymbol{\varepsilon}_0^*$, is

$$\langle\boldsymbol{S}\rangle \;=\; \frac{E^2}{2}\mathrm{Re}\;\boldsymbol{\varepsilon}_0\times\left(\frac{\boldsymbol{k}_0}{\mu_0\omega}\times\boldsymbol{\varepsilon}_0^*\right) = \hat{k}_0\frac{E^2}{2\mu_0 c}\mathrm{Re}\;\boldsymbol{\varepsilon}_0\cdot\boldsymbol{\varepsilon}_0^* = \hat{k}_0\frac{E^2}{2Z_0} \tag{748}$$

Dividing $dP/d\Omega$ by $E^2/2Z_0$ gives the differential cross section

$$\frac{d\sigma}{d\Omega} \;=\; \frac{q^4Z_0^2}{16\pi^2m^2c^2}|\hat{r}\times(\hat{r}\times\boldsymbol{\varepsilon}_0)|^2 = \left(\frac{q^2}{4\pi\epsilon_0\hbar c}\right)^2\left(\frac{\hbar}{mc}\right)^2|\hat{r}\times(\hat{r}\times\boldsymbol{\varepsilon}_0)|^2 \tag{749}$$

If we observe a particular polarization $\boldsymbol{\epsilon}$ in the scattered wave we replace $\hat{r}\times(\hat{r}\times\boldsymbol{\varepsilon}_0)\to \boldsymbol{\varepsilon}^*\cdot\hat{r}\times(\hat{r}\times\boldsymbol{\varepsilon}_0) = -\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0$ inside the absolute values:

$$\frac{d\sigma}{d\Omega} \;=\; \left(\frac{q^2}{4\pi\epsilon_0\hbar c}\right)^2\left(\frac{\hbar}{mc}\right)^2|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0|^2 = \frac{q^4}{e^4}\frac{m_e^2}{m^2}\left(\frac{\alpha\hbar}{m_e c}\right)^2|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0|^2 \tag{750}$$

which is just the Thomson cross section. Here $\alpha = e^2/4\pi\epsilon_0\hbar c \approx 1/137$ is the fine structure constant. The length $r_c = \alpha\hbar/m_e c$ is actually independent of $\hbar$ and is sometimes called the classical radius of the electron. The Bohr radius is the length $a_0 = \hbar/m_e c\alpha \approx .5 \times 10^{-8}$cm. we see that r_c is $\alpha^2 = O(10^{-4})$ smaller than the Bohr radius. If polarization is not measured we use

$$\sum_\lambda \varepsilon^i_\lambda \varepsilon^{j*}_\lambda = \delta_{ij} - \hat{k}^i\hat{k}^j \tag{751}$$

to obtain

$$\frac{d\sigma}{d\Omega} = \frac{q^4}{e^4}\left(\frac{\alpha\hbar}{mc}\right)^2 (1 - |\hat{k}\cdot\boldsymbol{\varepsilon}_0|^2) \tag{752}$$

If in addition the beam is unpolarized we find

$$\frac{d\sigma}{d\Omega} = \frac{q^4}{e^4}\left(\frac{\alpha\hbar}{mc}\right)^2 \frac{1+\cos^2\theta}{2} \tag{753}$$

This simple long wavelength calculation of the cross section can be extended to any target for which the dipole moment induced by a uniform (since $ka \ll 1$) field is known. In our electrostatic study of a dielectric sphere of radius a we found the induced electric dipole moment

$$\boldsymbol{p} = 4\pi\epsilon_0 a^3 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\boldsymbol{E} \tag{754}$$

Since an em wave contains a magnetic field as well as an electric field, a magnetic moment is also induced.

$$\boldsymbol{m} = 4\pi a^3 \frac{\mu - \mu_0}{\mu + 2\mu_0}\boldsymbol{H} \tag{755}$$

The limits $\epsilon \to \infty$ and $\mu \to 0$ give the induced electric and magnetic moments of a perfectly conducting sphere:

$$\boldsymbol{p} = 4\pi\epsilon_0 a^3\boldsymbol{E}, \qquad \boldsymbol{m} = -2\pi a^3\boldsymbol{H}, \qquad \text{Perfectly Conducting Sphere} \tag{756}$$

In the radiation zone expressions for the fields $\boldsymbol{p}$ and $\boldsymbol{m}$ come together in the combination

$$\begin{aligned}\frac{i}{\omega}\int d^3r \boldsymbol{J} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} &\sim (\boldsymbol{p} - \frac{1}{c}\hat{k}\times\boldsymbol{m} + \cdots \qquad) \\ &\to 4\pi a^3\epsilon_0\left(\boldsymbol{E}_0 + \frac{1}{2c\epsilon_0}\hat{k}\times\boldsymbol{H}_0\right) = 4\pi a^3\epsilon_0\left(\boldsymbol{E}_0 + \frac{1}{2}\hat{k}\times(\hat{k}_0\times\boldsymbol{E}_0)\right)\end{aligned}$$

We can obtain the cross section for a perfectly conducting sphere from that of a particle through the substitutions $q^2/m\omega^2 \to 4\pi\epsilon_0 a^3$ and $\boldsymbol{\varepsilon}_0 \to \boldsymbol{\varepsilon}_0 + \hat{k}\times(\hat{k}_0\times\boldsymbol{\varepsilon}_0)/2$:

$$\frac{d\sigma}{d\Omega} \approx a^6k^4|\boldsymbol{\varepsilon}^*\cdot\hat{k}\times(\hat{k}\times(\boldsymbol{\varepsilon}_0 + \frac{1}{2}\hat{k}\times(\hat{k}_0\times\boldsymbol{\varepsilon}_0)))|^2$$

$$\approx \quad a^6k^4|\boldsymbol{\varepsilon}^*\cdot(\boldsymbol{\varepsilon}_0+\frac{1}{2}\hat{k}\times(\hat{k}_0\times\boldsymbol{\varepsilon}_0))|^2 = a^6k^4\left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0-\frac{1}{2}(\hat{k}\times\boldsymbol{\varepsilon}^*)\cdot(\hat{k}_0\times\boldsymbol{\varepsilon}_0)\right|^2$$

$$\approx \quad a^6k^4\left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\left(1-\frac{1}{2}\cos\theta\right)+\frac{1}{2}\hat{k}_0\cdot\boldsymbol{\varepsilon}^*\hat{k}\cdot\boldsymbol{\varepsilon}_0\right|^2 \qquad (757)$$

The angular dependence of this expression is quite involved. But it is easy to see that scattering is maximal in the backward direction: at $\theta=\pi$ the cross section is nine times its value at $\theta=0$. This asymmetry is due to the interference between the electric and magnetic dipole contributions to the amplitude. The $k^4=\omega^4/c^2$ frequency dependence shows that red light is much less scattered than blue light and is the essential reason the sky is blue and sunsets are red. This same frequency dependence is a typical feature of long wavelength scattering of bound systems of finite size: the size a is needed for dimensional reasons.

Scattering off a collection of objects located at various positions $\boldsymbol{r}_k$ can be easily handled in this approximation. The total fields are obtained by adding the contributions from each object. The incident field at kth location will be a common amplitude times the phase factor $e^{i\boldsymbol{k}_0\cdot\boldsymbol{r}_k}$. Also the field in the radiation zone from the kth object has an additional phase factor $e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_k}$. If all the scatterers are otherwise identical and multiple scattering is neglected (valid for a dilute enough gas) the cross section from many scatterers is just that of a single scatterer times the factor

$$F(\boldsymbol{q}) \quad = \quad \left|\sum_k e^{i\boldsymbol{q}\cdot\boldsymbol{r}_k}\right|^2 \qquad (758)$$

where $\boldsymbol{q}=\boldsymbol{k}_0-\boldsymbol{k}$. There are two opposite extremes in the evaluation of F. First, if the scatterer locations are random, the cross terms in the absolute square have random phases and so average to zero. Then $F=N$ the total number of particles. The other extreme is if the locations are regularly spaced (as in a crystal). Suppose for example that they are equally spaced on a line with spacing $\boldsymbol{L}$. Then we have

$$\left|\sum_{k=0}^{N-1}e^{i\boldsymbol{q}\cdot\boldsymbol{L}k}\right|^2=\left|\frac{1-e^{iN\boldsymbol{q}\cdot\boldsymbol{L}}}{1-e^{i\boldsymbol{q}\cdot\boldsymbol{L}}}\right|^2=\frac{\sin^2(N\boldsymbol{q}\cdot\boldsymbol{L}/2)}{\sin^2(\boldsymbol{q}\cdot\boldsymbol{L}/2)} \qquad (759)$$

which for large N is strongly peaked in the forward direction ($\boldsymbol{q}=0$), where it is of order $O(N^2)$.

12.2 General Formulation of Scattering

Suppose we were able to solve the full set of Maxwell equations and particle equations of motion exactly. How would we use the solution to calculate the differential cross section? The answer is that we would find the solution which satisfies the boundary conditions appropriate to the scattering process. These boundary conditions are that at large $\boldsymbol{r}$ well outside the target each field be a linear combination of a plane wave, representing the incident beam and outgoing radial waves of the form e^{ikr}/r. We lose no information by taking the incident wave in the positive z-direction. Then, assuming all fields have time dependence $e^{-i\omega t}$, the fields should have the asymptotic behavior

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{ikz} + \boldsymbol{F}\frac{e^{ikr}}{r} \tag{760}$$

$$\boldsymbol{H} = \frac{1}{Z_0}(\hat{z}\times\boldsymbol{E}_0)e^{ikz} + \frac{e^{ikr}}{r}\frac{\hat{k}\times\boldsymbol{F}}{Z_0} \tag{761}$$

The square of the coefficient of e^{ikr}/r enters the radiated power while the square of the coefficient of e^{ikz} enters the incident flux:

$$\frac{dP}{d\Omega} = \frac{1}{2Z_0}\boldsymbol{F}^*\cdot\boldsymbol{F}, \qquad \text{Flux} = \frac{1}{2Z_0}\boldsymbol{E}^*\cdot\boldsymbol{E} \tag{762}$$

$$\frac{d\sigma}{d\Omega} = \frac{\boldsymbol{F}^*\cdot\boldsymbol{F}}{\boldsymbol{E}^*\cdot\boldsymbol{E}} \tag{763}$$

It is then convenient to define the scattering amplitude $\boldsymbol{f}$ as the $\boldsymbol{F}$ associated with the normalized incident electric field $\boldsymbol{E}_0 = \boldsymbol{\varepsilon}_0$, $\boldsymbol{\varepsilon}_0^*\cdot\boldsymbol{\varepsilon}_0 = 1$:

$$\boldsymbol{E} \sim \boldsymbol{\varepsilon}_0 e^{ikz} + \frac{e^{ikr}}{r}\boldsymbol{f}, \qquad \frac{d\sigma}{d\Omega} = |\boldsymbol{\varepsilon}^*\cdot\boldsymbol{f}|^2 \tag{764}$$

12.3 The Born Approximation

When the interaction of the incident beam with the target is sufficiently weak, a perturbative calculation can be attempted. The Born approximation is the lowest order approximation to the scattering amplitude in such an approach. Our discussion will be restricted to targets characterized by dielectric and magnetic properties, $\boldsymbol{D}\neq\epsilon_0\boldsymbol{E}$ and $\boldsymbol{B}\neq\mu_0\boldsymbol{H}$ over a localized region. We also assume that all fields have the same harmonic dependence $e^{-i\omega t}$. Then we can rearrange Maxwell's equations,

$$\begin{aligned}
-i\omega\boldsymbol{D} &= \nabla\times\boldsymbol{H} = \nabla\times\frac{\boldsymbol{B}}{\mu_0} + \nabla\times\left(\boldsymbol{H}-\frac{\boldsymbol{B}}{\mu_0}\right)\\
&= \nabla\times\frac{\nabla\times\boldsymbol{E}}{i\omega\mu_0} + \nabla\times\left(\boldsymbol{H}-\frac{\boldsymbol{B}}{\mu_0}\right)\\
&= \nabla\times\frac{\nabla\times\boldsymbol{D}}{i\omega\epsilon_0\mu_0} + \nabla\times\frac{\nabla\times(\epsilon_0\boldsymbol{E}-\boldsymbol{D})}{i\omega\epsilon_0\mu_0} + \nabla\times\left(\boldsymbol{H}-\frac{\boldsymbol{B}}{\mu_0}\right)\\
(-\nabla^2-k^2)\boldsymbol{D} &= -\nabla\times(\nabla\times(\epsilon_0\boldsymbol{E}-\boldsymbol{D})) - i\omega\epsilon_0\nabla\times(\mu_0\boldsymbol{H}-\boldsymbol{B})
\end{aligned} \tag{765}$$

where $k=\omega\epsilon_0\mu_0=\omega/c$. We can then use the Green function for the Helmholtz equation to write an integral equation with scattering boundary conditions:

$$\boldsymbol{D} = \boldsymbol{D}_0 e^{ikz} - \int d^3x'\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\left[\nabla\times(\nabla\times(\epsilon_0\boldsymbol{E}-\boldsymbol{D})) + i\omega\epsilon_0\nabla\times(\mu_0\boldsymbol{H}-\boldsymbol{B})\right] \tag{766}$$

$$\sim \boldsymbol{D}_0 e^{ikz} - \frac{e^{ikr}}{r}\int\frac{d^3x'}{4\pi}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'}\left[\nabla\times(\nabla\times(\epsilon_0\boldsymbol{E}-\boldsymbol{D})) + i\omega\epsilon_0\nabla\times(\mu_0\boldsymbol{H}+\boldsymbol{B})\right]$$

$$\sim \boldsymbol{D}_0 e^{ikz} + \frac{e^{ikr}}{r}\int\frac{d^3x'}{4\pi}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'}\left[\boldsymbol{k}\times(\boldsymbol{k}\times(\epsilon_0\boldsymbol{E}-\boldsymbol{D})) + \omega\epsilon_0\boldsymbol{k}\times(\mu_0\boldsymbol{H}-\boldsymbol{B})\right] \tag{767}$$

with a similar equation for $\boldsymbol{B}$. If $\boldsymbol{D}_0$ is normalized to $\boldsymbol{\varepsilon}_0$ the coefficient of e^{ikr}/r is the scattering amplitude $\boldsymbol{f}$.

$$\boldsymbol{f} = \int \frac{d^3x'}{4\pi} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'} \left[\boldsymbol{k}\times\boldsymbol{k}\times(\epsilon_0\boldsymbol{E}-\boldsymbol{D}) + \omega\epsilon_0\boldsymbol{k}\times(\mu_0\boldsymbol{H}-\boldsymbol{B})\right] \quad (768)$$

The Born approximation is to write

$$\boldsymbol{D}-\epsilon_0\boldsymbol{E} = (\epsilon-\epsilon_0)\boldsymbol{E} \approx (\epsilon-\epsilon_0)\boldsymbol{E}_0 e^{ikz} = (\epsilon_r-1)\boldsymbol{\varepsilon}_0 e^{ikz}$$

and

$$\boldsymbol{B}-\mu_0\boldsymbol{H} = (\mu-\mu_0)\boldsymbol{H} \approx (\mu-\mu_0)\boldsymbol{H}_0 e^{ikz} = (\mu-\mu_0)\hat{z}\times\frac{\boldsymbol{\varepsilon}_0}{Z_0\epsilon_0}e^{ikz}$$

where $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are the fields of the incident plane wave.

$$\boldsymbol{f}_{\rm Born} = \int \frac{d^3x'}{4\pi} e^{ikz'-i\boldsymbol{k}\cdot\boldsymbol{r}'} \left[-\boldsymbol{k}\times(\boldsymbol{k}\times(\epsilon_r-1)\boldsymbol{\varepsilon}_0) - k\boldsymbol{k}\times((\mu_r-1)\hat{z}\times\boldsymbol{\varepsilon}_0)\right]$$

$$\boldsymbol{\varepsilon}^*\cdot\boldsymbol{f}_{\rm Born} = \frac{k^2}{4\pi}\int d^3x' e^{-i\boldsymbol{q}\cdot\boldsymbol{r}'} \left[(\epsilon_r-1)\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0 + (\mu_r-1)(\hat{k}\times\boldsymbol{\varepsilon}^*)\cdot(\hat{z}\times\boldsymbol{\varepsilon}_0)\right] \quad (769)$$

where $\boldsymbol{q} = \boldsymbol{k} - k\hat{z}$, which is analogous to the momentum transfer in quantum mechanics. The departures of ϵ, μ from their vacuum values plays the role of the potential in the quantum mechanical born approximation.

12.4 Scattering from a Perfectly Conducting Sphere

If the target of the scattering process is a fixed perfectly conducting sphere of radius R, the interaction of the target with the em field is completely subsumed in the boundary conditions $\boldsymbol{E}_\parallel(r=R)=0$ and $\boldsymbol{r}\cdot\boldsymbol{H}=0$. Expanding the fields in vector spherical harmonics

$$\boldsymbol{H} = \sum_{lm}\left[f_{lm}(kr)\boldsymbol{X}_{lm} - \frac{i}{k}\nabla\times(g_{lm}(kr)\boldsymbol{X}_{lm})\right] \quad (770)$$

$$\boldsymbol{E} = Z_0\sum_{lm}\left[\frac{i}{k}\nabla\times f_{lm}(kr)\boldsymbol{X}_{lm} + g_{lm}(kr)\boldsymbol{X}_{lm})\right] \quad (771)$$

we see that these boundary conditions amount to

$$g_{lm}(kR)=0, \qquad \frac{\partial}{\partial r}(rf_{lm}(kr))\Big|_{r=R} = 0 \quad (772)$$

To apply these conditions we need to expand the incident plane wave in vector spherical harmonics. Since the plane wave is finite everywhere both spherical Bessel functions must be $j_l(kr)$:

$$\boldsymbol{E}_0 = \boldsymbol{\varepsilon}_0 e^{ikz} = Z_0\sum_{lm}\left[\frac{i}{k}a^E_{0lm}\nabla\times j_l(kr)\boldsymbol{X}_{lm} + a^M_{0lm}j_l(kr)\boldsymbol{X}_{lm})\right] \quad (773)$$

$$\boldsymbol{H}_0 = \frac{\hat{z}\times\boldsymbol{\varepsilon}_0}{Z_0}e^{ikz} = \sum_{lm}\left[a^E_{0lm}j_l(kr)\boldsymbol{X}_{lm} - \frac{i}{k}a^M_{0lm}\nabla\times(j_l(kr)\boldsymbol{X}_{lm})\right] \quad (774)$$

Then, dotting both fields into $\boldsymbol{r}$, we find

$$Z_0 j_l(kr) a^M_{lm} = \frac{k}{\sqrt{l(l+1)}} \int d\Omega Y^*_{lm} \boldsymbol{r} \cdot (\hat{z} \times \boldsymbol{\varepsilon}_0) e^{ikz} \tag{775}$$

$$Z_0 j_l(kr) a^E_{lm} = \frac{-k}{\sqrt{l(l+1)}} \int d\Omega Y^*_{lm} \boldsymbol{r} \cdot \boldsymbol{\varepsilon}_0 e^{ikz} \tag{776}$$

the easiest way to do these integrals is to identify the spherical harmonic expansion of e^{ikz}, which we can get by taking the $r \to \infty$ and $\theta \to 0$ limit of

$$\begin{aligned}
\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} &= ik \sum_{l=0}^{\infty} h_l^{(1)}(kr_>) j_l(kr_<) \sum_{m=-l}^{l} Y_{lm}(\theta,\varphi) Y^*_{lm}(\theta',\varphi') \\
\frac{e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'}}{4\pi} &= i \sum_{l=0}^{\infty} (-i)^{l+1} j_l(kr') Y_{l0}(0,\varphi) Y^*_{l0}(\theta',\varphi')) \\
e^{i\boldsymbol{k}\cdot\boldsymbol{r}'} &= \sum_{l=0}^{\infty} i^l \sqrt{4\pi(2l+1)} j_l(kr') Y_{l0}(\theta',\varphi') \\
e^{ikz} &= e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l \sqrt{4\pi(2l+1)} j_l(kr) Y_{l0}(\theta,\varphi) \\
r\sin\theta e^{\pm i\varphi} e^{ikz} &= i\frac{e^{\pm i\varphi}}{k} \frac{\partial}{\partial\theta} e^{ikr\cos\theta} = \frac{i}{k} \sum_{l=0}^{\infty} i^l \sqrt{4\pi(2l+1)} j_l(kr) e^{\pm i\varphi} \frac{\partial}{\partial\theta} Y_{l0}(\theta,\varphi) \\
&= \pm\frac{i}{k} \sum_{l=0}^{\infty} i^l \sqrt{4\pi(2l+1)} j_l(kr) \sqrt{l(l+1)} Y_{l\pm1}
\end{aligned} \tag{777}$$

Here we used the fact that $L_\pm Y_{l0} = \pm e^{\pm i\varphi} \partial Y_{l0}/\partial\theta$ which is true because Y_{l0} is independent of φ. Next we write, with $\epsilon_{0\pm} = (\epsilon_{0x} \pm i\epsilon_{0y})$

$$\boldsymbol{r} \cdot \boldsymbol{\varepsilon}_0 = r\sin\theta(\epsilon_{0x}\cos\varphi + \epsilon_{0y}\sin\varphi) = \frac{1}{2} r\sin\theta(\epsilon_{0+}e^{-i\varphi} + \epsilon_{0-}e^{i\varphi}) \tag{778}$$

$$\boldsymbol{r} \cdot (\hat{z} \times \boldsymbol{\varepsilon}_0) = r\sin\theta(\epsilon_{0x}\sin\varphi - \epsilon_{0y}\cos\varphi) = \frac{i}{2} r\sin\theta(\epsilon_{0+}e^{-i\varphi} - \epsilon_{0-}e^{i\varphi}) \tag{779}$$

Then

$$Z_0 a^M_{0lm} = \frac{1}{2} i^l \sqrt{4\pi(2l+1)} \left(\epsilon_{0+}\delta_{m,-1} + \epsilon_{0-}\delta_{m,1}\right) \tag{780}$$

$$Z_0 a^E_{0lm} = \frac{i}{2} i^l \sqrt{4\pi(2l+1)} \left(\epsilon_{0+}\delta_{m,-1} - \epsilon_{0-}\delta_{m,1}\right) \tag{781}$$

$$\boldsymbol{H}_0 = \frac{1}{2Z_0} \sum_{lm} i^l \sqrt{4\pi(2l+1)} \Big[j_l(kr) i(\epsilon_{0+}\boldsymbol{X}_{l-1} - \epsilon_{0-}\boldsymbol{X}_{l1})$$

$$-\frac{i}{k}\nabla \times (j_l(kr)(\epsilon_{0+}\boldsymbol{X}_{l-1} + \epsilon_{0-}\boldsymbol{X}_{l1})\Big] \tag{782}$$

$$\boldsymbol{E}_0 = \frac{1}{2}\sum_{lm} i^l\sqrt{4\pi(2l+1)}\Big[\frac{i}{k}\nabla \times j_l(kr)i(\epsilon_{0+}\boldsymbol{X}_{l-1} - \epsilon_{0-}\boldsymbol{X}_{l1})$$

$$+j_l(kr)(\epsilon_{0+}\boldsymbol{X}_{l-1} + \epsilon_{0-}\boldsymbol{X}_{l1})\Big] \tag{783}$$

To get the total fields we add the scattered wave which has only outgoing waves, and so involves the first spherical Hankel functions:

$$\boldsymbol{H}_S = \sum_{lm}\left[a^E_{Slm}h^{(1)}_l(kr)\boldsymbol{X}_{lm} - \frac{i}{k}a^M_{Slm}\nabla \times (h^{(1)}_l(kr)\boldsymbol{X}_{lm})\right] \tag{784}$$

$$\boldsymbol{E}_S = Z_0\sum_{lm}\left[\frac{i}{k}a^E_{Slm}\nabla \times h^{(1)}_l(kr)\boldsymbol{X}_{lm} + a^M_{Slm}h^{(1)}_l(kr)\boldsymbol{X}_{lm})\right] \tag{785}$$

We read off the scattering amplitude as the coefficient of e^{ikr}/r in $\boldsymbol{E}$:

$$\boldsymbol{f} = \frac{Z_0}{k}\sum_{lm}(-i)^{l+1}\left[a^M_{Slm}\boldsymbol{X}_{lm} - a^E_{Slm}\hat{k}\times\boldsymbol{X}_{lm}\right] \tag{786}$$

Then we have

$$f_{lm}(kr) = a^E_{0lm}j_l(kr) + a^E_{Slm}h^{(1)}_l(kr) \equiv a^E_{0lm}\left[j_l(kr) + \frac{\beta_l}{2}h^{(1)}_l(kr)\right]$$

$$g_{lm}(kr) = a^M_{0lm}j_l(kr) + a^M_{Slm}h^{(1)}_l(kr) \equiv a^M_{0lm}\left[j_l(kr) + \frac{\alpha_l}{2}h^{(1)}_l(kr)\right] \tag{787}$$

and the boundary conditions then read

$$a^M_{0lm}j_l(kR) + a^M_{Slm}h^{(1)}_l(kR) = 0, \qquad \frac{a^M_{Slm}}{a^M_{0lm}} = -\frac{j_l(kR)}{h^{(1)}(kR)} \equiv \frac{\alpha_l}{2} \tag{788}$$

$$\frac{d}{dr}r(a^E_{0lm}j_l(kr) + a^E_{Slm}h^{(1)}_l(kr))\Big|_{r=R} = 0, \qquad \frac{a^E_{Slm}}{a^E_{0lm}} = -\frac{(rj_l)'(kR)}{(rh^{(1)}_l)'(kR)} \equiv \frac{\beta_l}{2} \tag{789}$$

The scattering amplitude is then

$$\boldsymbol{f} = \frac{Z_0}{2k}\sum_{lm}(-i)^{l+1}\left[\alpha_l a^M_{0lm}\boldsymbol{X}_{lm} - \beta_l a^E_{0lm}\hat{k}\times\boldsymbol{X}_{lm}\right]$$

$$= -i\frac{\sqrt{\pi}}{2k}\sum_{l=1}^{\infty}\sqrt{2l+1}\left[\alpha_l\left(\epsilon_{0+}\boldsymbol{X}_{l-1} + \epsilon_{0-}\boldsymbol{X}_{l1}\right) - i\beta_l\hat{k}\times\left(\epsilon_{0+}\boldsymbol{X}_{l-1} - \epsilon_{0-}\boldsymbol{X}_{l1}\right)\right] \tag{790}$$

$$\alpha_l = -\frac{2j_l(kR)}{h^{(1)}_l(kR)} = -1 - \frac{h^{(2)}_l(kR)}{h^{(1)}_l(kR)} \equiv e^{2i\delta_l} - 1 \tag{791}$$

$$\beta_l = -\frac{2(kRj_l(kR))'}{(kRh^{(1)}_l(kR))'} = -1 - \frac{(kRh^{(2)}_l(kR))'}{(kRh^{(1)}_l(kR))'} \equiv e^{2i\delta'_l} - 1 \tag{792}$$

where $\delta_l,\ \delta'_l$ are called the scattering phase shifts. The formula for $\boldsymbol{f}$ gives the exact scattering amplitude for a perfectly conducting sphere, albeit as an infinite sum. Notice that the sum over l starts at $l=1$ because the m values of the vector spherical harmonics are $m=\pm 1$.

As a check on the formula, let's take the long wavelength limit $kR \ll 1$ to compare with our earlier calculation. For this we need the small argument limits of the the spherical Bessel functions:

$$j_l(x) \sim \frac{x^l}{(2l+1)!!}, \qquad (xj_l(x))' \sim (l+1)\frac{x^l}{(2l+1)!!} \tag{793}$$

$$h_l^{(1)}(x) \sim -i\frac{(2l-1)!!}{x^{l+1}}, \qquad (xh_l^{(1)}(x))' \sim il\frac{(2l-1)!!}{x^{l+1}} \tag{794}$$

$$\alpha_l \sim -2i\frac{(kR)^{2l+1}}{(2l+1)!!(2l-1)!!}, \qquad \beta_l \sim 2i\frac{l+1}{l}\frac{(kR)^{2l+1}}{(2l+1)!!(2l-1)!!} \tag{795}$$

Thus the $l=1$ terms dominate the long wavelength limit, $\alpha_1 \sim -2i(kR)^3/3$ and $\beta_1 \sim 4i(kR)^3/3$.

$$\boldsymbol{f} = -\sqrt{\frac{\pi}{3}}k^2R^3\left[(\epsilon_{0+}\boldsymbol{X}_{1-1}+\epsilon_{0-}\boldsymbol{X}_{11}) + (2i)\hat{k}\times(\epsilon_{0+}\boldsymbol{X}_{1-1}-\epsilon_{0-}\boldsymbol{X}_{11})\right] \tag{796}$$

The vector spherical harmonics for $l=1$ can be evaluated

$$X^z_{1m} = \frac{m}{\sqrt{2}}Y_{1m}, \qquad X^x_{1m} \pm iX^y_{1m} = \frac{\sqrt{2-m(m\pm 1)}}{\sqrt{2}}Y_{1m\pm1} \tag{797}$$

With a little algebra one can work out the relations

$$\epsilon_{0+}\boldsymbol{X}_{1-1} - \epsilon_{0-}\boldsymbol{X}_{11} = -i\sqrt{\frac{3}{4\pi}}\hat{k}\times\boldsymbol{\varepsilon}_0 \tag{798}$$

$$\epsilon_{0+}\boldsymbol{X}_{1-1} + \epsilon_{0-}\boldsymbol{X}_{11} = -\sqrt{\frac{3}{4\pi}}\hat{k}\times(\hat{k}_0\times\boldsymbol{\varepsilon}_0) \tag{799}$$

$$\boldsymbol{f} = -\frac{k^2R^3}{2}\left[-\hat{k}\times(\hat{k}_0\times\boldsymbol{\varepsilon}_0) + 2\hat{k}\times(\hat{k}\times\boldsymbol{\varepsilon}_0)\right]$$

$$\boldsymbol{\varepsilon}^*\cdot\boldsymbol{f} = k^2R^3\left[-\frac{1}{2}(\hat{k}\times\boldsymbol{\varepsilon}^*)\cdot(\hat{k}_0\times\boldsymbol{\varepsilon}_0) + \boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right] \tag{800}$$

which precisely agrees with our direct long wavelength calculation. The first term is the magnetic and the second the electric dipole contributions.

12.5 Short wavelength approximation and diffraction

When the wavelength of the radiation is much shorter than the system size, we can bring in a different simplification. It is based on the idea that in the short wavelength limit geometrical optics becomes valid. In quantum mechanics this idea is expressed in the WKB approximation in which the wave mechanics goes over to particle mechanics and interference phenomena become negligible. For scattering of light it becomes a good approximation to think of the beam as a bundle of rays that follow definite trajectories. Classic diffraction theory has to do with the first corrections to the geometrical optics approximation in which some interference is taken into account.

We begin our discussion with diffraction. Imagine a screen with several apertures occupying the xy-plane. Light is incident on the screen from the region $z < 0$ and is observed at large $z > 0$. In zeroth approximation $ka \to \infty$ the light rays are either completely reflected or pass through the apertures undeflected casting sharp shadow images of the apertures on a distant screen. If ka is large but finite the wave nature of light comes into play. The components of the fields satisfy the Helmholtz equation. An approximate way to think about this is to imagine that the fields in the region $z > 0$ are sourced by the fields in the apertures, which can be approximated by the fields of the incident wave in the apertures. To implement this idea introduce a Green function $G(\boldsymbol{r}, \boldsymbol{r}')$ for the Helmholtz equation, $(-\nabla^2 - k^2)G = \delta(\boldsymbol{r} - \boldsymbol{r}')$ and write out Green's Theorem in the half-space $z > 0$ for a field component solving $(-\nabla^2 - k^2)F = 0$

$$F(\boldsymbol{r}) \;=\; -\oint_S dS' \left(F(\boldsymbol{r}')\boldsymbol{n}\cdot\nabla' G(\boldsymbol{r}', \boldsymbol{r}) - G\boldsymbol{n}\cdot\nabla' F\right) \tag{801}$$

Here we take the surface to be the xy-plane closed by a hemisphere of infinite radius in the upper half space. Since G and F both will have large r behavior e^{ikr}/r the leading behavior of the normal derivative on the hemisphere of either F or G is simply a factor of ik times itself, so this leading $1/r^2$ term cancels between the two terms in the integrand, which then has large r behavior $1/r^3$. Thus the contribution of the hemisphere to the right side vanishes, leaving only the contribution of the xy-plane. On the xy-plane $\boldsymbol{n} = -\hat{z}$ so we obtain

$$F(\boldsymbol{r}) \;=\; \int dx'dy' \left(F\frac{\partial G}{\partial z'} - G\frac{\partial F}{\partial z'}\right)\Big|_{z'=0} \tag{802}$$

this exact formula can then be used to implement plausible approximations. Intuitively we would like the only contribution to come from the apertures, where the values of $F, \partial_{z'}F$ are approximated by the their values in the incident field.

A self-consistent way to implement this approximation is to choose G to satisfy either Neumann of Dirichlet boundary conditions on the xy plane. Then the above formula can be written in two ways

$$F(\boldsymbol{r}) \;=\; -\int dx'dy' G_N \frac{\partial F}{\partial z'}\Big|_{z'=0}, \qquad F(\boldsymbol{r}) = \int dx'dy' F\frac{\partial G_D}{\partial z'}\Big|_{z'=0} \tag{803}$$

$$G_{\substack{N\\D}} \;=\; \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \pm \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'_P|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'_P|} \tag{804}$$

$$G_N \;\to\; \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{2\pi|\boldsymbol{r}-\boldsymbol{r}'|}, \qquad \partial_{z'}G_D \to -\left(ikz - \frac{z}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{2\pi|\boldsymbol{r}-\boldsymbol{r}'|^2} \tag{805}$$

where $\boldsymbol{r}_P = (x, y, -z)$. The limiting forms in the last line are for $z' = 0$. Then the approximation for diffraction is

$$F(\boldsymbol{r}) \approx -\int_{\text{apertures}} dx'dy' \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{2\pi|\boldsymbol{r}-\boldsymbol{r}'|} \frac{\partial F_0}{\partial z'}\Big|_{z'=0} \tag{806}$$

$$F(\boldsymbol{r}) \approx -\int_{\text{apertures}} dx'dy' F_0 \left(ikz - \frac{z}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{2\pi|\boldsymbol{r}-\boldsymbol{r}'|^2}\Big|_{z'=0} \tag{807}$$

where F_0 is the corresponding field in the incident beam. An earlier less consistent version of this approximation chooses $G = G_0 = e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}/4\pi|\boldsymbol{r}-\boldsymbol{r}'|$, the empty space version and restricts the original expression to the apertures.

$$F(\boldsymbol{r}) = \int_{\text{apertures}} dx'dy' \left(F_0 \frac{\partial G_0}{\partial z'} - G_0 \frac{\partial F_0}{\partial z'}\right) \tag{808}$$

This last form is less consistent because it implicitly assumes both F and $\partial_{z'}F$ vanish on the screen off the apertures, which is impossible unless $F = 0$ everywhere. In the first two versions one requires $F = 0$ or $\partial_{z'}F = 0$ but not both. However, in the regime where the approximation is valid, namely in the far forward region, all three forms agree.

What we have described so far treats all field components independently, whereas we know that Maxwell's equations imply relations among the field components, only two of which are independent. A proper vectorial formulation is complicated except for the case of a perfectly conducting screen with apertures. In that case a remarkably simple relation emerges due to Smythe. The derivation, which is subtle and involved, is sketched in Jackson. The upshot is the exact formula for the diffracted electric field in the region $z > 0$

$$\boldsymbol{E}_{\text{diff}} = \nabla \times \int dx'dy' \hat{z} \times \boldsymbol{E} \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{2\pi|\boldsymbol{r}-\boldsymbol{r}'|} = \nabla \times \int_{\text{apertures}} dx'dy' \; \hat{z} \times \boldsymbol{E} \; \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{2\pi|\boldsymbol{r}-\boldsymbol{r}'|} \tag{809}$$

by perfect conductor boundary conditions. The approximate form then just inserts the incident field in the integral

$$\boldsymbol{E}_{\text{diff}} \approx \nabla \times \int_{\text{apertures}} dx'dy' \; \hat{z} \times \boldsymbol{E}_0 \; \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{2\pi|\boldsymbol{r}-\boldsymbol{r}'|} \tag{810}$$

$$\sim i\boldsymbol{k} \times \frac{e^{ikr}}{2\pi r} \int_{\text{apertures}} dx'dy' \; \hat{z} \times \boldsymbol{E}_0 \; e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'} \tag{811}$$

You will study an application of this formula in one of the new homework problems.

12.6 Short Wavelength Scattering

In order to apply geometrical optics concepts to a general scattering process including polarization dependence, we need a vectorial version of Green's theorem. A convenient form is given by

$$\boldsymbol{E}(\boldsymbol{r}) = -\oint_S dS' \left[i\omega(\boldsymbol{n}' \times \boldsymbol{B})G + (\boldsymbol{n}' \times \boldsymbol{E}) \times \nabla' G + (\boldsymbol{n}' \cdot \boldsymbol{E})\nabla' G\right] \tag{812}$$

To prove this one writes the right side as a volume integral of the expression with $\boldsymbol{n}' \to \nabla'$ and reduces that expression using the sourceless Maxwell equations:

$$\begin{aligned}
&i\omega(\nabla' \times \boldsymbol{B})G + (\nabla' \times \boldsymbol{E}) \times \nabla' G + (\nabla' \cdot \boldsymbol{E})\nabla' G \\
&= \nabla' \times (G\nabla' \times \boldsymbol{E}) + \nabla'_i(\boldsymbol{E}\nabla'_i G) - \nabla'(\boldsymbol{E} \cdot \nabla' G) + \nabla'_i(E_i \nabla' G) \\
&= \nabla'_i(G\nabla' E_i) - \nabla'_i(G\nabla'_i \boldsymbol{E}) + \nabla'_i(\boldsymbol{E}\nabla'_i G) - \nabla'(\boldsymbol{E} \cdot \nabla' G) + \nabla'_i(E_i \nabla' G) \\
&= G\nabla'^2 \boldsymbol{E} + \boldsymbol{E}\nabla'^2 G = -\boldsymbol{E}\delta(\boldsymbol{r} - \boldsymbol{r}')
\end{aligned} \tag{813}$$

Integrating both sides over the region bounded by S completes the proof.

We next choose S to bound the region between a surface S_1 that encloses the target system and is just outside it and a large spherical surface at infinity (for which $\boldsymbol{n}' = \hat{r}$). Since (**??**) applies for any solution of the sourceless Maxwell equations, it applies separately to the total field $\boldsymbol{E} = \boldsymbol{E}_0 + \boldsymbol{E}_s$ and the scattered field $\boldsymbol{E}_s$ when we discuss scattering. For the scattered field we now argue that the contribution of the spherical surface at infinity vanishes. Since both $\boldsymbol{E}_s$ and G have large r behavior e^{ikr}/r, the $1/r^2$ behavior of the integrand at infinity comes from replacing $\nabla' \to ik\boldsymbol{n}'$:

$$\begin{aligned}
i\omega(\boldsymbol{n}' \times \boldsymbol{B}_s)G + (\boldsymbol{n}' \times \boldsymbol{E}_s) \times \nabla' G + (\boldsymbol{n}' \cdot \boldsymbol{E}_s)\nabla' G &\to G(i\omega(\boldsymbol{n}' \times \boldsymbol{B}_s) + ik\boldsymbol{E}_s) \\
&\to O(r^{-3})
\end{aligned} \tag{814}$$

where we used

$$i\omega(\boldsymbol{n}' \times \boldsymbol{B}_s) = \boldsymbol{n}' \times (\nabla' \times \boldsymbol{E}_s) \to ik\boldsymbol{n}' \times (\boldsymbol{n}' \times \boldsymbol{E}_s) = ik\boldsymbol{n}'\boldsymbol{n}' \cdot \boldsymbol{E}_s - ik\boldsymbol{E}_s \to -ik\boldsymbol{E}_s + O(r^{-3})$$

since the the scattered electric field at large r is perpendicular to the radial direction. Thus we have

$$\boldsymbol{E}_s(\boldsymbol{r}) = -\oint_{S_1} dS' \left[i\omega(\boldsymbol{n}' \times \boldsymbol{B}_s)G + (\boldsymbol{n}' \times \boldsymbol{E}_s) \times \nabla' G + (\boldsymbol{n}' \cdot \boldsymbol{E}_s)\nabla' G\right] \tag{815}$$

Since the scattered field $\boldsymbol{E}_s$ has asymptotic large r behavior $\boldsymbol{f}e^{ikr}/r$, we can read off the scattering amplitude in terms of the surface integral over S_1 by taking the large r limit of the Green function

$$G \sim \frac{e^{ikr}}{4\pi r}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'}, \qquad \nabla' G \sim -i\boldsymbol{k}\frac{e^{ikr}}{4\pi r}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'} \tag{816}$$

$$\boldsymbol{f} = \frac{-i}{4\pi}\oint_{S_1} dS' \left[\omega(\boldsymbol{n}' \times \boldsymbol{B}_s) - (\boldsymbol{n}' \times \boldsymbol{E}_s) \times \boldsymbol{k} - \boldsymbol{k}(\boldsymbol{n}' \cdot \boldsymbol{E}_s)\right] e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'} \tag{817}$$

Since $\boldsymbol{k}\cdot\boldsymbol{f} = 0$, the last term in square brackets should after integration cancel the component from the first term parallel to $\boldsymbol{k}$. However this issue can be finessed by specifying a definite final polarization, which automatically removes all terms parallel to $\boldsymbol{k}$:

$$\boldsymbol{\varepsilon}^* \cdot \boldsymbol{f} = \frac{-i}{4\pi}\oint_{S_1} dS' \boldsymbol{\varepsilon}^* \cdot \left[\omega(\boldsymbol{n}' \times \boldsymbol{B}_s) + \boldsymbol{k} \times (\boldsymbol{n}' \times \boldsymbol{E}_s)\right] e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'} \tag{818}$$

This formula is exact, valid for all wavelengths. Turning now to the problem of short wavelength scattering, we consider an opaque target system that is large in all respects compared to the wavelength. Then first considering the geometrical optics description, we can say that the incident light illuminates the front of the object, leaving the back in total darkness. In the geometrical optics limit the boundary between the illuminated and shadow regions is sharp. In applying the above formula we can divide the surface S_1 into dark and light regions. In the dark region it is reasonable to approximate $\boldsymbol{E}_s \approx -\boldsymbol{E}_0$.

$$\begin{aligned}\boldsymbol{\varepsilon}^* \cdot \boldsymbol{f}_{\text{shadow}} &\approx \frac{i}{4\pi}\int_{\text{shadow}} dS'\boldsymbol{\varepsilon}^* \cdot \left[(\boldsymbol{n}' \times (\boldsymbol{k}_0 \times \boldsymbol{\varepsilon}_0) + \boldsymbol{k} \times (\boldsymbol{n}' \times \boldsymbol{\varepsilon}_0)\right] e^{i(\boldsymbol{k}_0-\boldsymbol{k})\cdot\boldsymbol{r}'} \\ &\approx \frac{i}{4\pi}\int_{\text{shadow}} dS'\boldsymbol{\varepsilon}^* \cdot \left[\boldsymbol{k}_0\boldsymbol{\varepsilon}_0 \cdot \boldsymbol{n}' - \boldsymbol{\varepsilon}_0(\boldsymbol{k}_0 + \boldsymbol{k}) \cdot \boldsymbol{n}' + \boldsymbol{n}'\boldsymbol{k} \cdot \boldsymbol{\varepsilon}_0\right] e^{i(\boldsymbol{k}_0-\boldsymbol{k})\cdot\boldsymbol{r}'}\end{aligned}$$

Since $kr, k_0 r \gg 1$ the integrand oscillates rapidly and is suppressed unless $\boldsymbol{k} \approx \boldsymbol{k}_0$. Thus we can set $\boldsymbol{k} = \boldsymbol{k}_0$ in the prefactors and the amplitude simplifies to

$$\boldsymbol{\varepsilon}^* \cdot \boldsymbol{f}_{\text{shadow}} \approx -\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0 \frac{i}{2\pi}\int_{\text{shadow}} dS'\boldsymbol{k}_0 \cdot \boldsymbol{n}' e^{i(\boldsymbol{k}_0-\boldsymbol{k})\cdot\boldsymbol{r}'} \tag{819}$$

Let $d^2x_\perp$ be an element of area perpendicular to $\boldsymbol{k}_0$. Then $dS'\boldsymbol{k}_0 \cdot \boldsymbol{n}' = -kd^2x$ (the sign coming because $\boldsymbol{n}'$ was the inward normal to S_1). Furthermore $\boldsymbol{r}' \cdot (\boldsymbol{k}_0 - \boldsymbol{k}) = z'k(1-\cos\theta) - \boldsymbol{k}_\perp \cdot \boldsymbol{r}_\perp \approx -\boldsymbol{k}_\perp \cdot \boldsymbol{r}_\perp + O(\theta^2)$. That is the integral is over the region projected by the shadow surface on the plane perpendicular to $\boldsymbol{k}_0$:

$$\boldsymbol{\varepsilon}^* \cdot \boldsymbol{f}_{\text{shadow}} \approx \boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0 \frac{ik}{2\pi}\int_{\text{shadow}} d^2x' e^{-i\boldsymbol{k}_\perp \cdot \boldsymbol{r}'_\perp} \tag{820}$$

If the projected area is a disk of radius R, the integral is

$$\int_{\text{shadow}} d^2x' e^{-i\boldsymbol{k}_\perp \cdot \boldsymbol{r}'_\perp} = \int_0^R \rho d\rho \int_0^{2\pi} d\varphi e^{-ik\rho\sin\theta\cos\varphi} \tag{821}$$

$$= 2\pi \int_0^R \rho d\rho J_0(k\rho\sin\theta) = 2\pi R^2 \frac{J_1(kR\sin\theta)}{kR\sin\theta} \tag{822}$$

$$\boldsymbol{\varepsilon}^* \cdot \boldsymbol{f}_{\text{shadow}} \approx ikR^2 \boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0 \frac{J_1(kR\sin\theta)}{kR\sin\theta} \tag{823}$$

The homework problem J, 10.16 shows that the part of the total cross section due to shadow scattering is the projected area, regardless of the shape of scatterer. In the case of the disc that is πR^2. The part of the scattering amplitude coming from the illuminated part of S_1 depends in detail on the nature of the surface.

12.7 The Optical Theorem

Conservation of energy in classical physics (and of probability in quantum physics) lead to an interesting useful connection between the total scattering cross section and the forward elastic scattering amplitude. We discuss this optical theorem in the context of classical electromagnetism. We assume harmonic time variation and take the time averaged Poynting vector to measure the energy flow.

$$\langle \boldsymbol{S} \rangle = \frac{1}{2}\mathrm{Re}\, \boldsymbol{E} \times \boldsymbol{H}^* \tag{824}$$

The total cross section is total power removed from the incident beam divided by the incident flux of energy. Power can be removed from the beam in two distinct ways: (1) the target can absorb energy and (2) the target can scatter energy out to infinity. to get expressions for these power losses we surround the target by a surface S_1. Let us define the outward normal at each point on the surface to be $\boldsymbol{n}$. We write the total fields as a sum of incident and scattered fields

$$\boldsymbol{E} = \boldsymbol{E}_0 + \boldsymbol{E}_s, \qquad \boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{H}_s \tag{825}$$

Then the total power scattered out to infinity is carried by the scattered fields:

$$P_s = \frac{1}{2}\mathrm{Re} \oint_{S_1} dS \boldsymbol{n} \cdot \boldsymbol{E}_s \times \boldsymbol{H}_s^* \tag{826}$$

The total power absorbed by the target is determined by the total fields

$$P_a = -\frac{1}{2}\mathrm{Re} \oint_{S_1} dS \boldsymbol{n} \cdot \boldsymbol{E} \times \boldsymbol{H}^* \tag{827}$$

And of course the incident beam is not affected by the target so it carries zero net power into the region bounded by S_1

$$\frac{1}{2}\mathrm{Re} \oint_{S_1} dS \boldsymbol{n} \cdot \boldsymbol{E}_0 \times \boldsymbol{H}_0^* = 0 \tag{828}$$

The total power removed from the beam is then

$$\begin{aligned} P &= \frac{1}{2}\mathrm{Re} \oint_{S_1} dS \boldsymbol{n} \cdot (\boldsymbol{E}_s \times \boldsymbol{H}_s^* - \boldsymbol{E} \times \boldsymbol{H}^*) \\ &= \frac{1}{2}\mathrm{Re} \oint_{S_1} dS \boldsymbol{n} \cdot (-\boldsymbol{E}_0 \times \boldsymbol{H}_0^* - \boldsymbol{E}_0 \times \boldsymbol{H}_s^* - \boldsymbol{E}_s \times \boldsymbol{H}_0^*) \\ &= -\frac{1}{2}\mathrm{Re} \oint_{S_1} dS \boldsymbol{n} \cdot (\boldsymbol{E}_0^* \times \boldsymbol{H}_s + \boldsymbol{E}_s \times \boldsymbol{H}_0^*) \end{aligned} \tag{829}$$

The final step is to parameterize the incident beam as a plane wave, which we normalize to unit field strength $E_0 = 1$:

$$\boldsymbol{E}_0 = \boldsymbol{\varepsilon}_0 e^{i\boldsymbol{k}_0 \cdot \boldsymbol{r}}, \qquad \boldsymbol{H}_0 = Z_0^{-1} \hat{k}_0 \times \boldsymbol{\varepsilon}_0 e^{i\boldsymbol{k}_0 \cdot \boldsymbol{r}} \tag{830}$$

With this normalization convention, our expressions for the "power" P have the units of watts/(volts/m)^2 or m$^2/\Omega$. Also the incident "flux" is just $1/2Z_0$! Plugging these expressions into the total power gives

$$\begin{aligned}
\sigma_{\rm total} &\equiv \frac{P}{1/2Z_0} = -{\rm Re}\oint_{S_1} dS\boldsymbol{n}\cdot(\boldsymbol{\varepsilon}_0^*\times c\mu_0\boldsymbol{H}_s + \boldsymbol{E}_s\times(\hat{k}_0\times\boldsymbol{\varepsilon}_0^*))e^{-i\boldsymbol{k}_0\cdot\boldsymbol{r}} \\
&= -{\rm Re}\oint_{S_1} dS\boldsymbol{n}\cdot(\boldsymbol{\varepsilon}_0^*\times c\boldsymbol{B}_s + \boldsymbol{E}_s\times(\hat{k}_0\times\boldsymbol{\varepsilon}_0^*))e^{-i\boldsymbol{k}_0\cdot\boldsymbol{r}} \\
&= {\rm Re}\oint_{S_1} dS\boldsymbol{\varepsilon}_0^*\cdot(\boldsymbol{n}\times c\boldsymbol{B}_s + \hat{k}_0\times(\boldsymbol{n}\times\boldsymbol{E}_s))e^{-i\boldsymbol{k}_0\cdot\boldsymbol{r}} \qquad (831)
\end{aligned}$$

Finally, recall our formula for the scattering amplitude

$$\boldsymbol{\varepsilon}^*\cdot\boldsymbol{f} = \frac{-i}{4\pi}\oint_{S_1} dS'\boldsymbol{\varepsilon}^*\cdot\left[\omega(\boldsymbol{n}'\times\boldsymbol{B}_s) + \boldsymbol{k}\times(\boldsymbol{n}'\times\boldsymbol{E}_s)\right]e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'} \qquad (832)$$

where we remember that the normal in this formula is inwardly directed w.r.t. S_1. Changing the sign and evaluating this formula in the forward elastic direction ($\boldsymbol{k} = \boldsymbol{k}_0$ and $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0$) we find

$$\begin{aligned}
\boldsymbol{\varepsilon}_0^*\cdot\boldsymbol{f}|_{\boldsymbol{k}=\boldsymbol{k}_0} &= \frac{ik}{4\pi}\oint_{S_1} dS\boldsymbol{\varepsilon}_0^*\cdot\left[c(\boldsymbol{n}\times\boldsymbol{B}_s) + \hat{k}_0\times(\boldsymbol{n}\times\boldsymbol{E}_s)\right]e^{-i\boldsymbol{k}_0\cdot\boldsymbol{r}'} \\
{\rm Im}\ \boldsymbol{\varepsilon}_0^*\cdot\boldsymbol{f}|_{\boldsymbol{k}=\boldsymbol{k}_0} &= \frac{k}{4\pi}{\rm Re}\oint_{S_1} dS\boldsymbol{\varepsilon}_0^*\cdot\left[c(\boldsymbol{n}\times\boldsymbol{B}_s) + \hat{k}_0\times(\boldsymbol{n}\times\boldsymbol{E}_s)\right]e^{-i\boldsymbol{k}_0\cdot\boldsymbol{r}'} \qquad (833)
\end{aligned}$$

The optical theorem immediately follows

$$\sigma_{\rm total} = \frac{4\pi}{k}{\rm Im}\ \boldsymbol{\varepsilon}_0^*\cdot\boldsymbol{f}|_{\boldsymbol{k}=\boldsymbol{k}_0} \qquad (834)$$

Notice that the right side is the imaginary part of the completely forward amplitude: the initial and final states being exactly the same. It is also important to appreciate that the right side involves the exact scattering amplitude. For example the Born approximation (first order perturbation theory) is real, with the imaginary part appearing only in corrections to the approximation. In this regard, notice that a direct calculation of the left side involves the square of scattering amplitudes, whereas the right side is linear. Thus if $\boldsymbol{f} = O(g)$ the left side of the optical theorem is $O(g^2)$ and it follows that the $O(g)$ contribution to the right side vanish, implying that the $O(g)$ contribution to the forward scattering amplitude must be real.

13 Energy Loss and Cherenkov Radiation

A charged particle moving through matter will lose energy through its interaction with the atoms in the material. This is a very broad subject, covered in some detail by Chapter 13 of Jackson. Here, of all the different aspects of the subject, I will single out just one: the radiation by a particle passing through matter at a speed v that exceeds the phase velocity of light in the material $v > c/n$ where $n > 1$ is the index of refraction. This is Cherenkov radiation, named after the physicist who discovered it. I will follow the treatment of Landau and Lifshitz, *Electrodynamics of Continuous Matter*, Chapter 12.

To set the stage, we first discuss the energy loss from a particle with any velocity, so we can appreciate what is special about $v > c/n$. We need to solve Maxwell's equations with the free charge and current densities of a point particle moving at velocity $\boldsymbol{v}$:

$$\begin{aligned} \rho(\boldsymbol{r},t) &= Q\delta(\boldsymbol{r}-\boldsymbol{v}t) = \int \frac{d^3k}{(2\pi)^3}(Qe^{-i\boldsymbol{k}\cdot\boldsymbol{v}t})e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \\ \boldsymbol{J}(\boldsymbol{r},t) &= Q\boldsymbol{v}\delta(\boldsymbol{r}-\boldsymbol{v}t) = \int \frac{d^3k}{(2\pi)^3}(Q\boldsymbol{v}e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t})e^{i\boldsymbol{k}\cdot\boldsymbol{r}}. \end{aligned} \tag{835}$$

We see that the Fourier transforms of the sources have harmonic time dependence $e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}$! This means that if we Fourier transform the Maxwell equations the fields in $\boldsymbol{k}$ space will also have this time dependence:

$$\boldsymbol{H} = \int \frac{d^3k}{(2\pi)^3}\boldsymbol{H}_{\boldsymbol{k}}e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{836}$$

$$\boldsymbol{E} = \int \frac{d^3k}{(2\pi)^3}\boldsymbol{E}_{\boldsymbol{k}}e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{837}$$

We will assume that $\mu = \mu_0$ and $\epsilon(\omega) > \epsilon_0$. Since the fields oscillate with frequency $\omega = \boldsymbol{k}\cdot\boldsymbol{v}$, we use the dielectric constant $\epsilon(\boldsymbol{k}\cdot\boldsymbol{v})$ in the Maxwell equations for $\boldsymbol{H}_{\boldsymbol{k}}, \boldsymbol{E}_{\boldsymbol{k}}$:

$$\boldsymbol{k}\cdot\boldsymbol{H}_{\boldsymbol{k}} = 0, \qquad i\boldsymbol{k}\times\boldsymbol{H}_{\boldsymbol{k}} + i\boldsymbol{k}\cdot\boldsymbol{v}\epsilon\boldsymbol{E}_{\boldsymbol{k}} = Q\boldsymbol{v} \tag{838}$$

$$i\boldsymbol{k}\cdot\epsilon\boldsymbol{E} = Q, \qquad i\boldsymbol{k}\times\boldsymbol{E}_{\boldsymbol{k}} = i\boldsymbol{k}\cdot\boldsymbol{v}\mu_0\boldsymbol{H}_{\boldsymbol{k}} \tag{839}$$

$$\boldsymbol{H}_{\boldsymbol{k}} = \frac{\boldsymbol{k}\times\boldsymbol{E}_{\boldsymbol{k}}}{\boldsymbol{k}\cdot\boldsymbol{v}\mu_0}, \qquad \boldsymbol{E}_{\boldsymbol{k}} = -i\frac{Q}{\epsilon}\frac{\boldsymbol{k}-\boldsymbol{v}\cdot\boldsymbol{k}\mu_0\epsilon\boldsymbol{v}}{\boldsymbol{k}^2-(\boldsymbol{k}\cdot\boldsymbol{v})^2\mu_0\epsilon} \tag{840}$$

Choose coordinates so $\boldsymbol{v} = v\hat{z}$ and calculate the electric field in coordinate space

$$\boldsymbol{E} = -i\int\frac{d^2k_\perp dk_z}{(2\pi)^3}\frac{Q}{\epsilon}\frac{\boldsymbol{k}_\perp + \hat{z}k_z(1-v^2\mu_0\epsilon)}{\boldsymbol{k}_\perp^2 + k_z^2(1-v^2\mu_0\epsilon)}e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{v}t)} \tag{841}$$

This field exerts a force on the particle given by

$$\boldsymbol{F} = Q\boldsymbol{E}(\boldsymbol{v}t) = -iQ^2\hat{z}\int\frac{d^2k_\perp dk_z}{(2\pi)^3}\frac{1}{\epsilon}\frac{k_z(1-v^2\mu_0\epsilon)}{\boldsymbol{k}_\perp^2 + k_z^2(1-v^2\mu_0\epsilon)} \tag{842}$$

$$= -iQ^2\hat{z}\int\frac{\pi dk_\perp^2 dk_z}{(2\pi)^3}\frac{1}{\epsilon}\frac{k_z(1-v^2\mu_0\epsilon)}{\boldsymbol{k}_\perp^2 + k_z^2(1-v^2\mu_0\epsilon)} \tag{843}$$

This force should of course be 0 for motion in the vacuum $\epsilon = \epsilon_0$. It will be if we interpret the (log divergent) k_z integral as the limit of the integral over the domain $-K < k_z < K$ as $K \to \infty$. In the material $\epsilon(k_z v)$ and we can get a nonzero result. If $v < \sqrt{\mu_0 \epsilon} = c/n$, a nonzero integral will arise only if $\text{Im}\epsilon \neq 0$. This is because the real part is even and the imaginary part is odd.

We now see the what is special about the case $v > c/n$. In that case the denominator in the integrand vanishes in the range of integration over $\boldsymbol{k}_\perp$, with the possibility of a nonzero integral with real ϵ, i.e. a transparent material. To interpret the integral near this zero we recall that ϵ will always have a small positive (negative) imaginary part for $\omega > 0$ ($\omega < 0$). In the first case the denominator has a small negative imaginary part, so we take the contribution of a small semi-circle contour into the lower half plane, which yields a contribution to $\boldsymbol{F}$ of

$$-iQ^2 \hat{z}(\pi i) \frac{\pi dk_z}{(2\pi)^3} \frac{1}{\epsilon} k_z (1 - v^2 \mu_0 \epsilon) = -\frac{Q^2}{8\pi} \hat{z} dk_z \frac{|k_z|}{\epsilon} (v^2 \mu_0 \epsilon - 1) \tag{844}$$

the contribution for negative k_z doubles this to give

$$d\boldsymbol{F} = -\frac{Q^2}{4\pi} \hat{z} dk_z \frac{|k_z|}{\epsilon} (v^2 \mu_0 \epsilon - 1) = -\frac{Q^2}{4\pi\epsilon_0} \hat{z} \frac{\omega d\omega}{c^2} \left(1 - \frac{c^2}{n^2 v^2}\right) \tag{845}$$

This force is opposite to the direction of $\boldsymbol{v}$, so its magnitude gives the energy loss per unit length due to Cherenkov radiation, which is a measure of the intensity of the radiation.

Since the entire contribution comes from the wave number for which the denominator of the integrand vanishes, we conclude that the radiation wave vector satisfies

$$0 = \mathbf{k}^2 - k_z^2 \frac{v^2 n^2}{c^2} = \boldsymbol{k}^2 \left(1 - \frac{v^2 n^2}{c^2} \cos^2\theta\right) \tag{846}$$

$$\cos\theta = \frac{c}{vn(\omega)} \tag{847}$$

Here θ is the angle between the Cherenkov radiation wave vector and the velocity of the charged particle. The radiation comes out on a cone of opening angle 2θ. This feature makes Cherenkov radiation into a tool to measure velocities of particles: measuring the opening angle of the cone of radiation and its frequency (to determine $n(\omega)$), one can then calculate the speed of the particle. Furthermore, from the direction of the electric field it is clear that the polarization of the radiation is in the plane determined by $\boldsymbol{k}$ and $\boldsymbol{v}$.

14 Radiation from a particle in relativistic motion

In most applications so far we have assumed that the systems of particles producing radiation are in nonrelativistic motion. Here we remove this restriction, at least for the case of a single charged point particle.

14.1 Liénard-Wiechert Potentials and Fields

Let the position of the particle at time t be $\boldsymbol{r}(t)$. Then the charge and current densities are given by

$$\rho(\boldsymbol{r},t) = q\delta(\boldsymbol{r}-\boldsymbol{r}(t)), \qquad \boldsymbol{J}(\boldsymbol{r},t) = q\dot{\boldsymbol{r}}(t)\delta(\boldsymbol{r}-\boldsymbol{r}(t)) \tag{848}$$

In Lorenz gauge the vector potential satisfies the inhomogeneous wave equation, which we solve in terms of the retarded Green function

$$-\partial^2 A_\mu(\boldsymbol{r},t) = \left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A_\mu = \mu_0 J_\mu(\boldsymbol{r},t) \tag{849}$$

$$A_\mu(\boldsymbol{r},t) = \mu_0 \int d^3x' dt' G_r(\boldsymbol{r},t;\boldsymbol{r}',t') J_\mu(\boldsymbol{r}',t') \tag{850}$$

$$G_r(\boldsymbol{r},t;\boldsymbol{r}',t') = \frac{\delta(t-t'-|\boldsymbol{r}-\boldsymbol{r}'|/c)}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \tag{851}$$

Plugging the current for a point particle into these formulas leads to the remarkably simple Lienard-Wiechert potentials

$$\phi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{(1-\boldsymbol{\beta}(t')\cdot\boldsymbol{n}(t'))|\boldsymbol{r}-\boldsymbol{r}(t')|}\bigg|_{t'=t-|\boldsymbol{r}-\boldsymbol{r}'|/c} \tag{852}$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\frac{q\dot{\boldsymbol{r}}(t')}{(1-\boldsymbol{\beta}(t')\cdot\boldsymbol{n}(t'))|\boldsymbol{r}-\boldsymbol{r}(t')|}\bigg|_{t'=t-|\boldsymbol{r}-\boldsymbol{r}'|/c} = \epsilon_0\mu_0\dot{\boldsymbol{r}}(t')\phi(\boldsymbol{r},t) \tag{853}$$

where we have introduced $\boldsymbol{\beta}(t') \equiv \dot{\boldsymbol{r}}(t')/c$ and $\boldsymbol{n}(t') \equiv (\boldsymbol{r}-\boldsymbol{r}(t'))/|\boldsymbol{r}-\boldsymbol{r}(t')|$. The factors $(1-\boldsymbol{\beta}(t')\cdot\boldsymbol{n}(t'))$ come from the integral of the $\delta(t-t'-|\boldsymbol{r}-\boldsymbol{r}(t')|/c)$ over t': recall that $\delta(f(x)) = \sum \delta(x-z_i)/|f'(z_i)|$ where z_i are the zeroes of f. The simplicity of these formulas is somewhat deceptive, since one only knows $t'(\boldsymbol{r},t)$ implicitly as the solution of $t' = t - |\boldsymbol{r}-\boldsymbol{r}(t')|/c$.

To calculate the fields from the potentials we have to take into account the dependence of t' on both t and $\boldsymbol{r}$:

$$\frac{\partial t'}{\partial t} = 1 + \frac{\partial t'}{\partial t}\frac{\dot{\boldsymbol{r}}(t')}{c}\cdot\boldsymbol{n}, \qquad \frac{\partial t'}{\partial t} = \frac{1}{1-\boldsymbol{\beta}(t')\cdot\boldsymbol{n}(t')} \tag{854}$$

$$\nabla t' = -\frac{\boldsymbol{n}}{c} + \nabla t'\frac{\dot{\boldsymbol{r}}(t')}{c}\cdot\boldsymbol{n}, \qquad \nabla t' = -\frac{\boldsymbol{n}}{c}\frac{1}{1-\boldsymbol{\beta}(t')\cdot\boldsymbol{n}(t')} \tag{855}$$

Armed with these results it suffices to calculate the derivatives of the potentials w.r.t. $\boldsymbol{r}, t'$:

$$\begin{aligned}
\frac{\partial\phi}{\partial t'} &= \frac{q}{4\pi\epsilon_0}\frac{\partial}{\partial t'}[|\boldsymbol{r}-\boldsymbol{r}(t')| - \boldsymbol{\beta}(t')\cdot(\boldsymbol{r}-\boldsymbol{r}(t'))]^{-1} \\
&= \frac{-q}{4\pi\epsilon_0}\frac{(\boldsymbol{\beta}-\boldsymbol{n})\cdot\boldsymbol{\beta}c - \dot{\boldsymbol{\beta}}\cdot(\boldsymbol{r}-\boldsymbol{r}(t'))}{[|\boldsymbol{r}-\boldsymbol{r}(t')| - \boldsymbol{\beta}(t')\cdot(\boldsymbol{r}-\boldsymbol{r}(t')]^2} = -q\frac{(\boldsymbol{\beta}-\boldsymbol{n})\cdot\boldsymbol{\beta}c - \dot{\boldsymbol{\beta}}\cdot(\boldsymbol{r}-\boldsymbol{r}(t'))}{4\pi\epsilon_0|\boldsymbol{r}-\boldsymbol{r}(t')|^2[1-\boldsymbol{\beta}\cdot\boldsymbol{n}]^2} \qquad (856) \\
\nabla\phi|_{t'} &= \frac{-q}{4\pi\epsilon_0}\frac{\boldsymbol{n}(t')-\boldsymbol{\beta}(t')}{[|\boldsymbol{r}-\boldsymbol{r}(t')| - \boldsymbol{\beta}(t')\cdot(\boldsymbol{r}-\boldsymbol{r}(t')]^2} = \frac{-q}{4\pi\epsilon_0}\frac{\boldsymbol{n}(t')-\boldsymbol{\beta}(t')}{|\boldsymbol{r}-\boldsymbol{r}(t')|^2[1-\boldsymbol{\beta}\cdot\boldsymbol{n}]^2} \qquad (857) \\
\boldsymbol{E} &= -\nabla\phi|_t - \frac{\partial\boldsymbol{A}}{\partial t} = -\nabla\phi|_{t'} - \nabla t'\frac{\partial\phi}{\partial t'} - \frac{\partial t'}{\partial t}\frac{\partial\boldsymbol{A}}{\partial t'} \\
&= -\nabla\phi|_{t'} - \nabla t'\frac{\partial\phi}{\partial t'} - \frac{1}{c^2}\frac{\partial t'}{\partial t}\left(\ddot{\boldsymbol{r}}\phi + \dot{\boldsymbol{r}}\frac{\partial\phi}{\partial t'}\right) \\
&= \frac{q}{4\pi\epsilon_0}\Big[\frac{\boldsymbol{n}(t')-\boldsymbol{\beta}(t')}{|\boldsymbol{r}-\boldsymbol{r}(t')|^2[1-\boldsymbol{\beta}\cdot\boldsymbol{n}]^2} - (\boldsymbol{n}-\boldsymbol{\beta})\frac{(\boldsymbol{\beta}-\boldsymbol{n})\cdot\boldsymbol{\beta}c - \dot{\boldsymbol{\beta}}\cdot(\boldsymbol{r}-\boldsymbol{r}(t'))}{c|\boldsymbol{r}-\boldsymbol{r}(t')|^2[1-\boldsymbol{\beta}\cdot\boldsymbol{n}]^3} \\
&\quad -\frac{\dot{\boldsymbol{\beta}}}{c}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}(t')|[1-\boldsymbol{\beta}\cdot\boldsymbol{n}]^2}\Big] \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{[\boldsymbol{n}-\boldsymbol{\beta}](1-\beta^2)}{|\boldsymbol{r}-\boldsymbol{r}(t')|^2[1-\boldsymbol{\beta}\cdot\boldsymbol{n}]^3} + \frac{1}{c}\frac{\boldsymbol{n}\times[(\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{|\boldsymbol{r}-\boldsymbol{r}(t')|[1-\boldsymbol{\beta}\cdot\boldsymbol{n}]^3}\right] \qquad (858) \\
\boldsymbol{B} &= \nabla\times\boldsymbol{A} = \epsilon_0\mu_0\left[\nabla t'\times\left(\ddot{\boldsymbol{r}}\phi + \dot{\boldsymbol{r}}\frac{\partial\phi}{\partial t'}\right) + \nabla\phi|_{t'}\times\dot{\boldsymbol{r}}\right] \\
&= \frac{q\mu_0}{4\pi}\Big[-\frac{\boldsymbol{n}}{c}\times\left(\frac{\ddot{\boldsymbol{r}}}{|\boldsymbol{r}-\boldsymbol{r}(t')|[1-\boldsymbol{\beta}\cdot\boldsymbol{n}]^2} + \dot{\boldsymbol{r}}\frac{(\boldsymbol{\beta}-\boldsymbol{n})\cdot\boldsymbol{\beta}c - \dot{\boldsymbol{\beta}}\cdot(\boldsymbol{r}-\boldsymbol{r}(t'))}{|\boldsymbol{r}-\boldsymbol{r}(t')|^2[1-\boldsymbol{\beta}\cdot\boldsymbol{n}]^2}\right) \\
&\quad -\frac{(\boldsymbol{n}-\boldsymbol{\beta})\times\boldsymbol{\beta}c}{|\boldsymbol{r}-\boldsymbol{r}(t')|^2[1-\boldsymbol{\beta}\cdot\boldsymbol{n}]^2}\Big] \\
&= \frac{q\mu_0}{4\pi}\boldsymbol{n}\times\left[-\frac{\dot{\boldsymbol{\beta}}[1-\boldsymbol{\beta}\cdot\boldsymbol{n}] + \boldsymbol{\beta}\boldsymbol{n}\cdot\dot{\boldsymbol{\beta}}}{|\boldsymbol{r}-\boldsymbol{r}(t')|[1-\boldsymbol{\beta}\cdot\boldsymbol{n}]^3} - \frac{\boldsymbol{\beta}c(1-\beta^2)}{|\boldsymbol{r}-\boldsymbol{r}(t')|^2[1-\boldsymbol{\beta}\cdot\boldsymbol{n}]^3}\right] \qquad (859) \\
&= \mu_0\epsilon_0 c\boldsymbol{n}\times\boldsymbol{E} = \frac{\mu_0}{Z_0}\boldsymbol{n}\times\boldsymbol{E} = \mu_0\boldsymbol{H} \qquad (860)
\end{aligned}$$

In these formulas it is understood that $\boldsymbol{\beta}$ and $\boldsymbol{n}$ are evaluated at the retarded time t'.

Radiation can be identified by taking $\boldsymbol{r}$ in the radiation zone $|\boldsymbol{r}| \gg |\boldsymbol{r}(t')|$ sufficiently large so the second terms (involving the acceleration) dominate. One can always do this when the motion of the charged particle is bounded. Even if it is not, one can always evaluate the instantaneous Poynting vector on a large sphere centered on $\boldsymbol{r}(t')$ which will then be an equal time surface, since the observation time $t = t' + |\boldsymbol{r}-\boldsymbol{r}(t')|/c$. When the radius of this sphere $R = |\boldsymbol{r}-\boldsymbol{r}(t')|$ is large enough the "acceleration terms" of the fields dominate: The Poynting vector is

$$\boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{H} = \frac{1}{Z_0}\boldsymbol{E}\times(\boldsymbol{n}\times\boldsymbol{E}) = \frac{1}{Z_0}(\boldsymbol{n}\boldsymbol{E}^2 - \boldsymbol{n}\cdot\boldsymbol{E}\boldsymbol{E}) \sim \frac{1}{Z_0}\boldsymbol{n}\boldsymbol{E}^2 \qquad (861)$$

where the last form uses the fact that the acceleration term of $\boldsymbol{E}$ is perpendicular to $\boldsymbol{n}$. Then the power per unit solid angle passing through this sphere is

$$R^2\boldsymbol{n}\cdot\boldsymbol{S} \sim \frac{q^2}{16\pi^2\epsilon_0 c}\frac{(\boldsymbol{n}\times[(\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}])^2}{(1-\boldsymbol{\beta}\cdot\boldsymbol{n})^6} \tag{862}$$

This gives the energy per unit t per solid angle emitted into a given solid angle. We must bear in mind that the quantities $\boldsymbol{\beta}(t')$and $\boldsymbol{n}(t')$ appearing on the right are all evaluated at the retarded time t'. If we want to get the total power by integrating over t', we should multiply it by a factor $dt/dt' = 1-\boldsymbol{\beta}\cdot\boldsymbol{n}$ to get the energy per unit t' which is the relevant quantity to integrate over the particle's trajectory to get the total radiated energy.

$$\frac{dP}{d\Omega}(t') = \frac{q^2}{e^2}\frac{\alpha\hbar}{4\pi}\frac{(\boldsymbol{n}\times[(\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}])^2}{(1-\boldsymbol{\beta}\cdot\boldsymbol{n})^5} \tag{863}$$

where we introduced the finite structure constant $\alpha = e^2/4\pi\epsilon_0\hbar c \approx 1/137$.

The virtue of this formula is that it gives the instantaneous radiated power with no restrictions on the speed of the radiating particle. When the particle is highly relativistic, the factor $(1-\boldsymbol{\beta}\cdot\boldsymbol{n})^5$ in the denominator gets very small in the forward direction, showing that the angular distribution is sharply peaked in the forward direction. For example if $\boldsymbol{\beta}$ is parallel to $\dot{\boldsymbol{\beta}}$, we have

$$\frac{dP}{d\Omega}(t') = \frac{q^2}{e^2}\frac{\alpha\hbar}{4\pi}\frac{\dot{\boldsymbol{\beta}}^2\sin^2\theta}{(1-\beta\cos\theta)^5} \tag{864}$$

where θ is the angle between $\boldsymbol{n}$ and $\boldsymbol{\beta}$.

Calculating the total radiated power by integrating the differential power over angles is a challenge with general $\boldsymbol{\beta}$, but in a Lorentz frame in which the particle is instantaneously at rest it is a piece of cake:

$$\frac{dP}{d\Omega}(t') \rightarrow \frac{q^2}{e^2}\frac{\alpha\hbar}{4\pi}(\boldsymbol{n}\times[\boldsymbol{n}\times\dot{\boldsymbol{\beta}}])^2 = \frac{q^2}{e^2}\frac{\alpha\hbar}{4\pi}\dot{\boldsymbol{\beta}}^2\sin^2\theta \tag{865}$$

$$P \rightarrow \frac{2}{3}\frac{q^2}{e^2}\alpha\hbar\dot{\boldsymbol{\beta}}^2, \qquad \beta \ll 1 \tag{866}$$

which gives the total radiated power for nonrelativistic motion, first calculated by Larmor. For general $\boldsymbol{\beta}$ Liénard obtained the result[9]

$$P = \frac{2}{3}\frac{q^2}{e^2}\alpha\hbar\gamma^6\left[\dot{\boldsymbol{\beta}}^2-(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2\right] = \frac{2}{3}\frac{q^2}{e^2}\alpha\hbar\left[\gamma^4\dot{\boldsymbol{\beta}}^2+\gamma^6(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2\right] \tag{870}$$

[9]The integral is elementary but tedious:

$$\begin{aligned}(\boldsymbol{n}\times[(\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}])^2 &= (\boldsymbol{n}-\boldsymbol{\beta})^2(\boldsymbol{n}\cdot\dot{\boldsymbol{\beta}})^2+(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^2\dot{\boldsymbol{\beta}}^2-2\boldsymbol{n}\cdot\dot{\boldsymbol{\beta}}(1-\boldsymbol{n}\cdot\boldsymbol{\beta})(\boldsymbol{n}-\boldsymbol{\beta})\cdot\dot{\boldsymbol{\beta}}\\ &= (\boldsymbol{\beta}^2-1)(\boldsymbol{n}\cdot\dot{\boldsymbol{\beta}})^2+(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^2\dot{\boldsymbol{\beta}}^2+2\boldsymbol{n}\cdot\dot{\boldsymbol{\beta}}(1-\boldsymbol{n}\cdot\boldsymbol{\beta})(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})\end{aligned} \tag{867}$$

For purposes of doing the angular integrals, choose $\boldsymbol{\beta} = \beta\hat{z}$ and $\dot{\boldsymbol{\beta}} = |\dot{\boldsymbol{\beta}}|(\hat{z}\cos\alpha+\hat{x}\sin\alpha)$. Then after the azimuthal integral $\boldsymbol{n}\cdot\dot{\boldsymbol{\beta}} \rightarrow |\dot{\boldsymbol{\beta}}|\cos\alpha\cos\theta$ and $(\boldsymbol{n}\cdot\dot{\boldsymbol{\beta}})^2 \rightarrow \dot{\boldsymbol{\beta}}^2(\sin^2\alpha\sin^2\theta+2\cos^2\alpha\cos^2\theta)$. Thus under the

One can rewrite the right side in terms of the time derivative of the four momentum dp^μ/dt as follows:

$$\frac{dp^0}{dt} = mc\frac{d\gamma}{dt} = \gamma^3 mc\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}, \qquad \frac{d\boldsymbol{p}}{dt} = \gamma mc\dot{\boldsymbol{\beta}} + \gamma^3 mc\boldsymbol{\beta}\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}} \tag{871}$$

$$\frac{dp^\mu}{dt}\frac{dp_\mu}{dt} = (\gamma mc)^2\left[(\dot{\boldsymbol{\beta}} + \gamma^2\boldsymbol{\beta}\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2 - \gamma^4(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2\right] = (\gamma mc)^2\left[\dot{\boldsymbol{\beta}}^2 + \gamma^2(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2\right] \tag{872}$$

$$P = \frac{2}{3}\frac{q^2}{e^2}\frac{\alpha\hbar}{m^2c^2}\gamma^2\frac{dp^\mu}{dt}\frac{dp_\mu}{dt} = \frac{2}{3}\frac{q^2}{e^2}\frac{\alpha\hbar}{m^2c^2}\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau} \tag{873}$$

which shows that P is in fact a Lorentz invariant.

14.2 Particle Accelerators

The radiation of highly relativistic particles has serious consequences for high energy physics which aims to accelerate particles to high momentum to probe the short distance structure of elementary particles as well as to produce new ones. First lets consider linear accelerators, so $\boldsymbol{v}$ is parallel to $\dot{\boldsymbol{v}}$. In this case

$$\left(\frac{dp^0}{dt}\right)^2 = \beta^2\left(\frac{d\boldsymbol{p}}{dt}\right)^2, \qquad \frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau} = \frac{1}{\gamma^2}\left(\frac{d\boldsymbol{p}}{d\tau}\right)^2 = \left(\frac{d\boldsymbol{p}}{dt}\right)^2 = \left(\frac{dE}{dx}\right)^2 \tag{874}$$

$$P = \frac{2}{3}\frac{q^2}{e^2}\frac{\alpha\hbar}{m^2c^2}\gamma^2\frac{dp^\mu}{dt}\frac{dp_\mu}{dt} = \frac{2}{3}\frac{q^2}{e^2}\frac{\alpha\hbar}{m^2c^2}\left(\frac{dE}{dx}\right)^2 \tag{875}$$

The power required to increase the energy of the particle is dE/dt so in comparison to this the radiated power,

$$\frac{P}{dE/dt} = \frac{2}{3}\frac{q^2}{e^2}\frac{\alpha\hbar}{m^2c^2v}\frac{dE}{dx} = \frac{2}{3}\frac{q^2}{e^2}\frac{\alpha^2 a_0}{mcv}\frac{dE}{dx} = O\left(\frac{10^{-14}\text{m}}{1\text{MeV}}\right)\frac{dE}{dx} \tag{876}$$

---

integral we can substitute

$$\begin{aligned}(\boldsymbol{n}\times[(\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}])^2 \;\to\; & \frac{\dot{\boldsymbol{\beta}}^2}{2}(\boldsymbol{\beta}^2-1)(\sin^2\alpha\sin^2\theta + 2\cos^2\alpha\cos^2\theta)\\ & +(1-\beta\cos\theta)^2\dot{\boldsymbol{\beta}}^2 + 2\dot{\boldsymbol{\beta}}^2\beta\cos^2\alpha\cos\theta(1-\beta\cos\theta)\\ \to\; & \dot{\boldsymbol{\beta}}^2(\boldsymbol{\beta}^2-1)\cos^2\theta + \frac{\dot{\boldsymbol{\beta}}^2}{2}\sin^2\alpha(1-3\cos^2\theta)(\boldsymbol{\beta}^2-1)\\ & +[1-\beta^2\cos^2\theta]\dot{\boldsymbol{\beta}}^2 - 2\dot{\boldsymbol{\beta}}^2\beta\sin^2\alpha\cos\theta(1-\beta\cos\theta)\\ \to\; & \dot{\boldsymbol{\beta}}^2(1-\cos^2\theta) + \frac{\dot{\boldsymbol{\beta}}^2\sin^2\alpha}{2}\left[(1-3\cos^2\theta)(\boldsymbol{\beta}^2-1) - 4\beta\cos\theta(1-\beta\cos\theta)\right]\\ \to\; & \dot{\boldsymbol{\beta}}^2(1-\cos^2\theta) + \frac{\dot{\boldsymbol{\beta}}^2\sin^2\alpha}{2}\left[(3+\beta^2)\cos^2\theta + \boldsymbol{\beta}^2 - 1 - 4\beta\cos\theta\right]\end{aligned} \tag{868}$$

Then

$$P = \frac{q^2\alpha\hbar}{2e^2}\dot{\boldsymbol{\beta}}^2\int_{-1}^{1}dz\frac{(1-z^2)+\sin^2\alpha\left[(3+\beta^2)z^2+\boldsymbol{\beta}^2-1-4\beta z\right]/2}{(1-\beta z)^5} = \frac{2q^2\alpha\hbar}{3e^2}\dot{\boldsymbol{\beta}}^2(1-\beta^2\sin^2\alpha) \tag{869}$$

for electrons. Linacs typically accelerate only 100 MeV/m so that for them radiation losses are completely negligible.

For circular accelerators (synchrotrons) the radiation coming from the changing direction of $\boldsymbol{\beta}$ is always present whether or not the speed of the particle is increasing. This energy loss must be continually replenished just to keep particles in the ring at fixed energy. In this case we have

$$\left(\frac{d\boldsymbol{p}}{dt}\right)^2 = \omega^2\boldsymbol{p}^2 \tag{877}$$

$$P = \frac{2}{3}\frac{q^2}{e^2}\frac{\alpha\hbar}{m^2c^2}\gamma^2\omega^2\boldsymbol{p}^2 = \frac{2}{3}\frac{q^2}{e^2}\alpha\hbar\gamma^4\omega^2\boldsymbol{\beta}^2 = \frac{2}{3}\frac{q^2}{e^2}\alpha\hbar\gamma^4\boldsymbol{\beta}^4\frac{c^2}{R^2} \tag{878}$$

$$= O\left(\frac{10^{-19}\mathrm{Wm}^2}{R^2}\gamma^4\right) \tag{879}$$

where R is the radius of hte ring. At the design energy of the LHC (7 TeV protons) $\gamma \approx 7000$. Whereas at LEP (60 GeV electrons) $\gamma \approx 120000$.

14.3 Charge in uniform motion

We next begin exploring the consequences of these exact expressions for the fields, starting with the case of uniform motion. For definiteness let's take the charge to move up the x-axis at constant speed βc. The expressions should then reduce to those obtained by a Lorentz transformation from the Coulomb electric field in the frame in which the particle is at rest: $\boldsymbol{E}' = q\boldsymbol{r}'/4\pi\epsilon_0 r'^3, \boldsymbol{B}' = 0$. To check this we need the ingredients of the formulas: $\boldsymbol{\beta} = \beta\hat{x}$ and $\boldsymbol{r}(t) = \hat{x}ct\beta$. The retarded time is determined by $c(t-t') = \sqrt{(x-\beta ct')^2 + y^2 + z^2}$. We get t' by solving a quadratic equation:

$$\begin{aligned}
ct' &= \frac{ct - \beta x - \sqrt{(ct-\beta x)^2 + (r^2 - c^2t^2)(1-\beta^2)}}{1-\beta^2} \\
&= \frac{ct - \beta x - \sqrt{(x-\beta ct)^2 + (y^2+z^2)(1-\beta^2)}}{1-\beta^2} \\
|\boldsymbol{r} - \boldsymbol{r}(t')|(1 - \boldsymbol{n}\cdot\boldsymbol{\beta}) &= |\boldsymbol{r} - \boldsymbol{r}(t')| - \boldsymbol{\beta}\cdot(\boldsymbol{r} - \hat{x}\beta ct') \\
&= ct - ct'(1-\beta^2) - \beta x = \sqrt{(x-\beta ct)^2 + (y^2+z^2)(1-\beta^2)} \\
&= r'\sqrt{1-\beta^2} \\
|\boldsymbol{r} - \boldsymbol{r}(t')|(\boldsymbol{n} - \boldsymbol{\beta}) &= \boldsymbol{r} - \hat{x}\beta ct' - \hat{x}\beta c(t-t') = \boldsymbol{r} - \hat{x}\beta ct
\end{aligned} \tag{880}$$

Putting these ingredients into the formula for the electric field gives

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon_0}\frac{\gamma(\boldsymbol{r} - \hat{x}\beta ct)}{[\gamma^2(x-\beta ct)^2 + y^2 + z^2)^{3/2}} = \frac{q}{4\pi\epsilon_0}\frac{\gamma(\boldsymbol{r} - \hat{x}\beta ct)}{r'^3} \tag{881}$$

where r' is the distance from the charge to the observation point as measured in the charge's rest frame. We now see that $E_x = E'_x$ and $E_{y,z} = \gamma E'_{y,z}$, precisely the desired Lorentz transform of the fields. From the identities

$$\boldsymbol{n} \times (\boldsymbol{n} - \boldsymbol{\beta}) = \boldsymbol{n} \times (-\boldsymbol{\beta}) = \boldsymbol{\beta} \times \boldsymbol{n} = \boldsymbol{\beta} \times (\boldsymbol{n} - \boldsymbol{\beta}) \tag{882}$$

we see that

$$\boldsymbol{B} = \frac{\mu_0}{Z_0}\boldsymbol{n} \times \boldsymbol{E} = \frac{\mu_0}{Z_0}\boldsymbol{\beta} \times \boldsymbol{E} = \frac{1}{c}\boldsymbol{\beta} \times \boldsymbol{E} \tag{883}$$

as required by the Lorentz transformation from the frame in which the particle is at rest. Of course, since the fields fall of as $1/r^2$, the Poynting vector falls off as $1/r^4$ and so there is no radiation of energy to infinity in nonaccelerating motion.

14.4 Charge moving with constant proper acceleration

Let us now consider a charge accelerating along the x-axis with constant acceleration in the particle's instantaneous rest frame. This problem has interesting implications for the equivalence principle of general relativity, because an observer moving with the particle will interpret the acceleration as a static gravitational field, and surely a particle held fixed in a static gravitational field should not radiate. We therefore focus on the nature of the radiation that can be produced by such a motion.

The trajectory of such a motion is given by $y(t) = z(t) = 0$ and

$$x(t) = \frac{c^2}{a}\sqrt{1 + \frac{a^2t^2}{c^2}}, \tag{884}$$

$$\boldsymbol{\beta} = \hat{x}\frac{at}{c}\left(1 + \frac{a^2t^2}{c^2}\right)^{-1/2}, \tag{885}$$

$$\dot{\boldsymbol{\beta}} = \hat{x}\frac{a}{c}\left(1 + \frac{a^2t^2}{c^2}\right)^{-3/2} = \frac{\boldsymbol{\beta}(1-\beta^2)}{t} \tag{886}$$

At early times $t \to -\infty$ the particle comes in from $x = +\infty$ at the speed of light, decelerates to rest at $t = 0$, reverses direction and accelerates toward $x = +\infty$ eventually attaining the speed of light once again.

Retarded time for $t = 0$

For simplicity we first work out the fields produced by this accelerating particle at $t = 0$. Then the retarded time is given by the equation

$$ct' = -|\boldsymbol{r} - \boldsymbol{r}(t')| = -\sqrt{(x - x(t'))^2 + y^2 + z^2}$$

$$x(t') = \frac{r^2 + c^4/a^2}{2x}, \qquad ct' = -\sqrt{\frac{(r^2 + c^4/a^2)^2}{4x^2} - \frac{c^4}{a^2}} \tag{887}$$

Then

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{[\boldsymbol{n} - \boldsymbol{\beta}](1-\beta^2)}{|\boldsymbol{r} - \boldsymbol{r}(t')|^2[1 - \boldsymbol{\beta}\cdot\boldsymbol{n}]^3} + \frac{1}{c}\frac{\boldsymbol{n}\boldsymbol{n}\cdot\dot{\boldsymbol{\beta}} - \dot{\boldsymbol{\beta}}}{|\boldsymbol{r} - \boldsymbol{r}(t')|[1 - \boldsymbol{\beta}\cdot\boldsymbol{n}]^3}\right], \qquad \boldsymbol{B} = \frac{\mu_0}{Z_0}\boldsymbol{n} \times \boldsymbol{E}$$

Now $\dot{\boldsymbol{\beta}}(t') = \boldsymbol{\beta}(1-\beta^2)/t' = -c\boldsymbol{\beta}(1-\beta^2)/|\boldsymbol{r}-\boldsymbol{r}(t')|$ by the definition of the retarded time. Thus the two terms in large square brackets combine very neatly into

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{n}(1-\beta^2)}{|\boldsymbol{r}-\boldsymbol{r}(t')|^2[1-\boldsymbol{\beta}\cdot\boldsymbol{n}]^2}, \qquad \boldsymbol{B}=0 \tag{888}$$

The vanishing of $\boldsymbol{B}$ follows immediately from the fact that $\boldsymbol{E}$ is parallel to $\boldsymbol{n}$. At $t=0$ the Poynting vector vanishes everywhere, implying that there is no flow flow of energy anywhere at that moment. Continuing the evaluation

$$\begin{aligned} |\boldsymbol{r}-\boldsymbol{r}(t')|[1-\boldsymbol{\beta}\cdot\boldsymbol{n}] &= |\boldsymbol{r}-\boldsymbol{r}(t')| - \beta(x-x(t')) = -\beta x \\ \frac{1-\beta^2}{|\boldsymbol{r}-\boldsymbol{r}(t')|^2[1-\boldsymbol{\beta}\cdot\boldsymbol{n}]^2} &= \frac{c^2}{a^2x^2t'^2} = \frac{4c^4}{a^2(r^2+c^4/a^2)^2-4x^2c^4} \end{aligned} \tag{889}$$

so we find

$$\begin{aligned} \boldsymbol{E} &= \frac{q}{4\pi\epsilon_0}\frac{4c^4\boldsymbol{n}}{a^2(r^2+c^4/a^2)^2-4x^2c^4} = \frac{q}{4\pi\epsilon_0}\frac{8|x|c^4(\boldsymbol{\rho}+\hat{x}(x-x(t'))}{a^2((r^2+c^4/a^2)^2-4x^2c^4/a^2)^{3/2}} \\ &= \frac{q}{\pi\epsilon_0}\frac{c^4(2|x|\boldsymbol{\rho}-\hat{x}(r^2-2x^2+c^4/a^2)}{a^2((r^2+c^4/a^2)^2-4x^2c^4/a^2)^{3/2}} \end{aligned} \tag{890}$$

In these expressions, we should understand that the fields using the retarded propagator are only non-zero in the domain $x \geq 0 - ct$, since we assume they are generated by a charge on a hyperbolic trajectory that keeps $x > 0$. For $t = 0$ this means that the fields are zero for $x < 0$.

Retarded time for $t \neq 0$ [Omit on a first reading)

For $t \neq 0$ the expressions for the fields are considerably more complicated. The retarded time t' is then determined as the solution of a quadratic equation

$$\begin{aligned} 2xx(t') &= r^2 - c^2t^2 + \frac{c^4}{a^2} + 2c^2tt' \equiv \xi^2(r,t) + 2c^2tt' \\ 0 &= (x^2-c^2t^2)t'^2 - \xi^2tt' + x^2\frac{c^2}{a^2} - \frac{\xi^4}{4c^2} \\ t' &= \frac{\xi^2t \pm \sqrt{\xi^4t^2-(x^2-c^2t^2)(4x^2c^2/a^2-\xi^4/c^2)}}{2(x^2-c^2t^2)} \\ ct' &= \frac{\xi^2ct \pm x\sqrt{\xi^4-4(x^2-c^2t^2)c^4/a^2}}{2(x^2-c^2t^2)} \end{aligned} \tag{891}$$

Of the two solutions, taking into account that the fields are nonzero only when $x + ct \geq 0$, the choice $-$ gives the correct retarded time

$$ct' = \frac{\xi^2ct - x\sqrt{\xi^4-4(x^2-c^2t^2)c^4/a^2}}{2(x^2-c^2t^2)} \tag{892}$$

$$
\begin{aligned}
|\boldsymbol{r}-\boldsymbol{r}(t')| &= c(t-t') = \frac{(2(x^2-c^2t^2)-\xi^2)ct + x\sqrt{\xi^4-4(x^2-c^2t^2)c^4/a^2}}{2(x^2-c^2t^2)} \\
x(t') &= \frac{\xi^2 x - ct\sqrt{\xi^4-4(x^2-c^2t^2)c^4/a^2}}{2(x^2-c^2t^2)} &(893)\\
\beta(t') &= \frac{ct'}{x(t')} = \frac{\xi^2 ct - x\sqrt{\xi^4-4(x^2-c^2t^2)c^4/a^2}}{\xi^2 x - ct\sqrt{\xi^4-4(x^2-c^2t^2)c^4/a^2}} &(894)\\
|\boldsymbol{r}-\boldsymbol{r}(t')|(1-\boldsymbol{n}\cdot\boldsymbol{\beta}) &= |\boldsymbol{r}-\boldsymbol{r}(t')| - \beta(x-x(t')) = ct-\beta(t')x \\
&= \frac{ct'}{x(t')} = \frac{(x^2-c^2t^2)\sqrt{\xi^4-4(x^2-c^2t^2)c^4/a^2}}{\xi^2 x - ct\sqrt{\xi^4-4(x^2-c^2t^2)c^4/a^2}} &(895)
\end{aligned}
$$

The last line shows the denominator of the Lienard-Wiechert potentials which then read:

$$
\begin{aligned}
\phi &= -\frac{1}{c}A_0 = \frac{q\theta(ct+x)}{4\pi\epsilon_0}\frac{\xi^2 x - ct\sqrt{\xi^4-4(x^2-c^2t^2)c^4/a^2}}{(x^2-c^2t^2)\sqrt{\xi^4-4(x^2-c^2t^2)c^4/a^2}} &(896)\\
A_x &= A_1 = \frac{q\theta(ct+x)}{4\pi\epsilon_0 c}\frac{\xi^2 ct - x\sqrt{\xi^4-4(x^2-c^2t^2)c^4/a^2}}{(x^2-c^2t^2)\sqrt{\xi^4-4(x^2-c^2t^2)c^4/a^2}}, \qquad A_y = A_z = 0 &(897)
\end{aligned}
$$

In these formulae we have supplied factors of $\theta(ct+x)$ to remind us that by construction the retarded potentials are zero when $ct < -x$

If we repeat the calculation of the potentials for a particle trajectory following the "image" hyperbola $x_-(t') = -x(t')$ we find the results

$$
\begin{aligned}
\phi^- &= -\frac{1}{c}A_0^- = \frac{q\theta(ct-x)}{4\pi\epsilon_0}\frac{-\xi^2 x - ct\sqrt{\xi^4-4(x^2-c^2t^2)c^4/a^2}}{(x^2-c^2t^2)\sqrt{\xi^4-4(x^2-c^2t^2)c^4/a^2}} &(898)\\
A_x^- &= A_1^- = \frac{q\theta(ct-x)}{4\pi\epsilon_0 c}\frac{-\xi^2 ct - x\sqrt{\xi^4-4(x^2-c^2t^2)c^4/a^2}}{(x^2-c^2t^2)\sqrt{\xi^4-4(x^2-c^2t^2)c^4/a^2}}, \qquad A_y^- = A_z^- = 0 &(899)
\end{aligned}
$$

What is interesting here is that in their common domain of support, i.e. $ct \geq |x|$ the sum of the original plus image potentials is a pure gauge:

$$
\begin{aligned}
A_0 + A_0^- &= \frac{q}{4\pi\epsilon_0}\frac{2t}{x^2-c^2t^2} = -\partial_0 \frac{q}{4\pi\epsilon_0 c}\ln(c^2t^2-x^2) &(900)\\
A_1 + A_1^- &= \frac{q}{4\pi\epsilon_0 c}\frac{-2x}{x^2-c^2t^2} = -\partial_1 \frac{q}{4\pi\epsilon_0 c}\ln(c^2t^2-x^2) &(901)
\end{aligned}
$$

What this means is that in the domain $ct > |x|$ the fields produced by the image charge are precisely the negative of the fields produced by the original charge.

Radiation and the Equivalence Principle

For the discussion of radiation, let's divide the xt plane into four sectors, following the approach of D. Boulware, Annals of Physics **124** (1980) 169-188. Region I is characterized by $x^2 > c^2t^2$ and $x > 0$. The hyperbolic trajectory lies entirely within region I. It is also the entire space-time accessible to an observer moving with the charged particle. Region II is characterized by $c^2t^2 > x^2$ and $t > 0$. The fields produced by the accelerating particle are nonzero only in $I \cup II$. Region III, the image of region I, is characterized by $x^2 > c^2t^2$ and $x < 0$. Every point in I is space-like separated from every point in Region III. The physics in III is completely decoupled from the physics in I. However charges in either region can produce fields in region II. Finally, region IV is the image of II, characterized by $c^2t^2 > x^2$ and $t < 0$. Because of retarded boundary conditions, the fields are strictly zero in region IV.

It is elementary that radiation emitted in region I will enter region II (see Peierls, Surprises in Theoretical Physics). Assume it is emitted at x_0, t_0, $x_0 > ct_0$ at an angle θ with the x-axis. Then at time $t = t_0 + R/c$ it will have reached $x = x_0 + R\cos\theta$. Then

$$ct - x = ct_0 + R - x_0 - R\cos\theta = ct_0 - x_0 + R(1-\cos\theta) \tag{902}$$

which will be in region II as soon as R is large enough. This argument also shows that the device of separating radiation fields from quasi-static fields by observations on a sufficiently large sphere cannot be applied strictly within region I (Pauli: no radiation zone can develop). The fact that fields in region II produced by the original trajectory in Region I, will be cancelled by fields produced by the image trajectory in region III, shows that it is consistent with Maxwell's equations to say that no radiation leaves region I. But since it is impossible to unambiguously identify radiation by observations in region I, it is fair to say there is no radiation at all: this interpretation is confirmed by the known absence of radiation reaction forces for hyperbolic motion. Thus there is no conflict with the equivalence principle.

On the other hand, if no fields are produced by charges in region III, radiation is definitely produced in Region II. As Boulware shows, the energy of this radiation comes not from radiation reaction on the particle, but from a cross term in the energy between the continuous field and a boundary delta function field. To calculate the radiation in region II we go to a large enough sphere so that the acceleration term of the Lienard-Wiechert field dominates:

$$\boldsymbol{E} \sim \frac{q}{4\pi\epsilon_0}\left[\frac{1}{c}\frac{\boldsymbol{n}\times[\boldsymbol{n}\times\dot{\boldsymbol{\beta}}]}{|\boldsymbol{r}-\boldsymbol{r}(t')|[1-\boldsymbol{\beta}\cdot\boldsymbol{n}]^3}\right] \tag{903}$$

$$\frac{dP}{d\Omega}(t') = \frac{q^2}{e^2}\frac{\alpha\hbar}{4\pi}\frac{\dot{\boldsymbol{\beta}}^2\sin^2\theta}{(1-\beta\cos\theta)^5} \tag{904}$$

$$P(t') = \frac{2}{3}\frac{q^2}{e^2}\alpha\hbar\gamma^6\dot{\boldsymbol{\beta}}^2 \tag{905}$$

For hyperbolic motion $1/\gamma^2 = 1-\beta^2 = (1+a^2t^2/c^2)^{-1}$ and $\dot\beta = (a/c\gamma^3)$, so

$$P(t') = \frac{2a^2}{3c^2}\frac{q^2}{e^2}\alpha\hbar \tag{906}$$

which is identical to the nonrelativistic Larmor result. Since it is independent of t' the total energy radiated is just PT where T is the time over which acceleration occurs.