

Electromagnetic Theory I

Solution Set 1

Due: 9 September 2020

This and all future homework will be posted on the course webpage:

<http://www.phys.ufl.edu/~thorn/homepage/eminfo.html>

In the homework assignments, I will refer to problems in Jackson by prefixing the problem number with J.

1. a) Solve for the motion of a point particle with charge q and mass m in constant homogeneous electric and magnetic fields that are perpendicular to each other. For definiteness take $\mathbf{B} = B\hat{z}$ parallel to the z -axis and $\mathbf{E} = E\hat{y}$ parallel to the y -axis. Take the initial condition that the particle is at rest at the origin of coordinates. You may assume that the speed of the particle remains much smaller than the speed of light throughout the motion.

Solution: Use Lorentz force law and $\mathbf{p} \approx m\mathbf{v}$ to obtain:

$$\frac{dv_x}{dt} = \frac{qB}{m}v_y, \quad \frac{dv_y}{dt} = -\frac{qB}{m}v_x + \frac{qE}{m} \quad (1)$$

$$\frac{d^2v_y}{dt^2} = -\frac{q^2B^2}{m^2}v_y, \quad \frac{d^2v_x}{dt^2} = -\frac{q^2B^2}{m^2}\left(v_x - \frac{E}{B}\right) \quad (2)$$

Clearly v_x, v_y harmonically oscillate at angular frequency $\omega = qB/m$ about the values $E/B, 0$ respectively. The solution for $v_x(t), v_y(t)$ with initial conditions $v_x(0) = v_y(0) = 0$ is

$$v_x(t) = \frac{dx}{dt} = \frac{E}{B} - \frac{E}{B}\cos\omega t, \quad v_y(t) = \frac{dy}{dt} = \frac{1}{\omega}\frac{dv_x}{dt} = \frac{E}{B}\sin\omega t \quad (3)$$

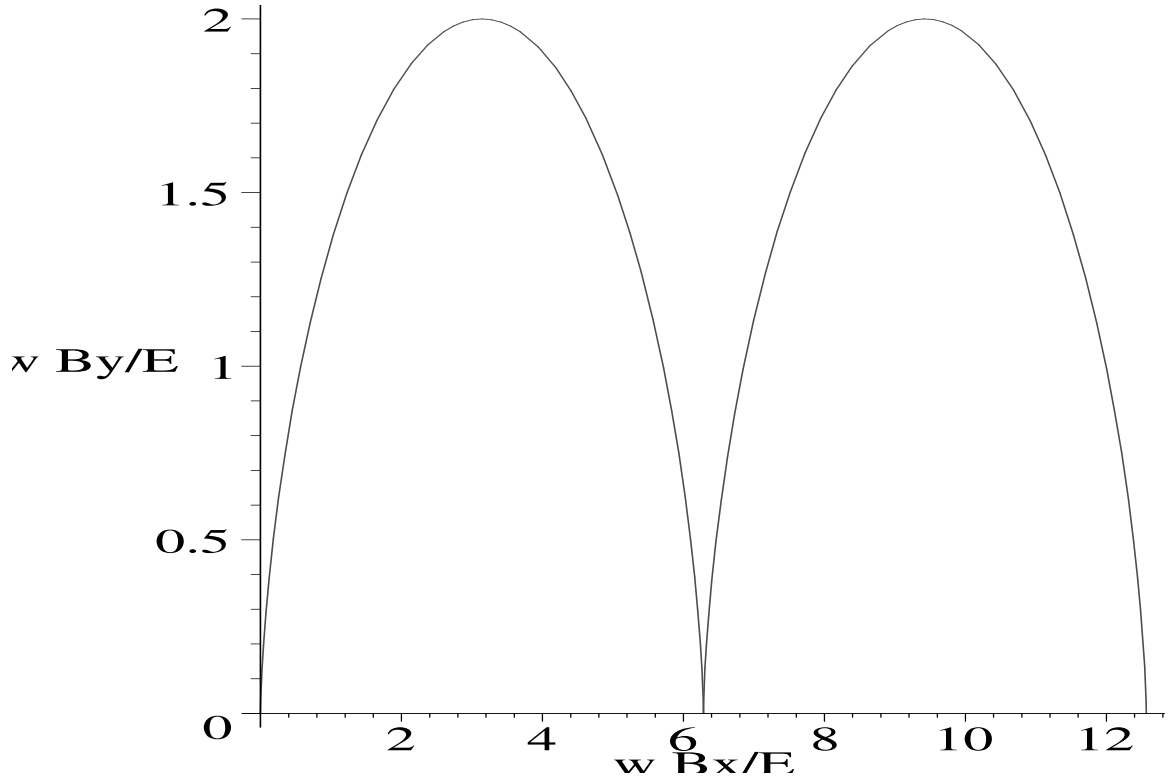
Integrating these with respect to t and fixing integration constants so that $x(0) = y(0) = 0$ gives

$$x(t) = \frac{E}{B}t - \frac{E}{B\omega}\sin\omega t, \quad y(t) = \frac{E}{B\omega}(1 - \cos\omega t) \quad (4)$$

- b) What condition on the fields must hold if the motion stays nonrelativistic throughout?

Solution: $v \ll c \Rightarrow E \ll cB$

- c) Draw the trajectory followed by the particle projected onto the x - y plane, indicating the coordinates of all the significant features of the motion.



2. J, Problem 1.3

Solution:

a) From the conditions $\rho = C\delta(r - R)$, where C is determined from

$$Q = \int d^3x \rho(r) = C \int d\Omega \int_0^\infty r^2 dr \delta(r - R) = 4\pi R^2 C \quad (5)$$

or $\rho = (Q/4\pi R^2)\delta(r - R)$. Here $d\Omega = \sin\theta d\theta d\varphi$ is the element of solid angle. The range of φ is $0 < \varphi < 2\pi$, and that of θ is $0 < \theta < \pi$, so $\int d\Omega = 4\pi$.

b) We call the radial cylindrical coordinate r_c instead of the customary ρ to avoid confusion with the charge density. Obviously $\rho = C\delta(r_c - b)$ and

$$\lambda = C \int_0^{2\pi} d\varphi \int_0^\infty r_c dr_c \delta(r_c - b) = 2\pi b C \quad (6)$$

so $\rho = (\lambda/2\pi b)\delta(r_c - b)$.

c) From the stated distribution, $\rho = C\Theta(R - r_c)\delta(z)$ where we take the disk in the plane $z = 0$ centered on the origin.

$$Q = C \int_0^{2\pi} d\varphi \int_0^\infty r_c dr_c \int_{-\infty}^\infty dz \Theta(R - r_c)\delta(z) = 2\pi(R^2/2)C = \pi R^2 C \quad (7)$$

so $\rho = (Q/\pi R^2)\Theta(R - r_c)\delta(z)$. Here $\Theta(x)$, defined to be 0 for $x < 0$ and 1 for $x > 0$, is the standard step function (also called the Heaviside function).

d) In spherical coordinates, $z = r \cos \theta$ and $r_c = r \sin \theta$. then from c)

$$\rho = (Q/\pi R^2)\Theta(R - r \sin \theta)\delta(r \cos \theta) = (Q/\pi r R^2)\Theta(R - r)\delta(\cos \theta) \quad (8)$$

since the delta function sets $\cos \theta = 0$ so that $\sin \theta \rightarrow 1$.

3. A total charge Q is distributed inside a sphere of radius R with an isotropic charge density $\rho(r)$.

a) Use Gauss' law to determine the electric field everywhere, both inside and outside the sphere. Your answer will involve an integral over r .

Solution: By spherical symmetry $\mathbf{E}(\mathbf{r}) = E(r)\hat{r}$, and Gauss's Law reads $E(r) = Q_{\leq r}/4\pi\epsilon_0 r^2$. For a general $\rho(r)$ confined to a sphere of radius R , $Q_{\leq r} = Q$ for $r \geq R$. For $r < R$ we can only write

$$Q_{\leq r} = \int_0^r dr' 4\pi r'^2 \rho(r'). \quad (9)$$

then

$$E(r) = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2} & r \geq R \\ \frac{1}{\epsilon_0 r^2} \int_0^r dr' r'^2 \rho(r') & r < R \end{cases} \quad (10)$$

b) Specialize to a charge density of the form $\rho(r) = A(R^2 - r^2)$. Determine A in terms of Q and R , and obtain an explicit expression for the electric field everywhere.

Solution: We first evaluate the integral

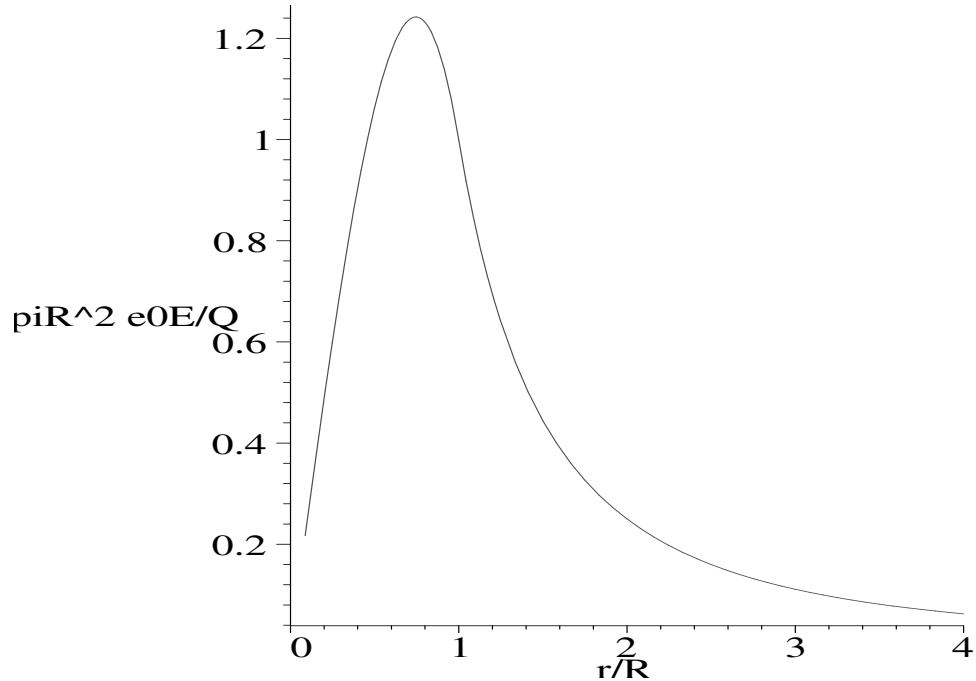
$$\int_0^r dr' r'^2 A(R^2 - r'^2) = A \left(\frac{r^3 R^2}{3} - \frac{r^5}{5} \right) \rightarrow A \frac{2R^5}{15} \quad (11)$$

for $r = R$. Setting this equal to $Q/4\pi$ gives $A = 15Q/8\pi R^5$. Then part a) gives

$$E(r) = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2} & r \geq R \\ \frac{Q}{8\pi\epsilon_0} \left(\frac{5r}{R^3} - \frac{3r^3}{R^5} \right) & r < R \end{cases} \quad (12)$$

c) Plot the magnitude of the electric field $E(r)$ as a function of r for the result of part b).

Solution:



4. We will be using SI units throughout this course. However, in Heaviside-Lorentz (H-L) units (see section 1.3 of the lecture notes) Maxwell's equations gain simplicity and clarity.

a) Show that the Heaviside-Lorentz unit of charge has dimensions of $(\text{Energy})^{1/2} \times (\text{Length})^{1/2}$.

Solution: From $\nabla \cdot \mathbf{E} = \rho$ in H-L units (Gauss' Law) it follows that charge has the units of electric field times length squared. Since electric field has units of force/charge, it follows that charge squared has units of force times length squared or energy times length.

b) Find the numerical value of the electron's charge \hat{e} in Heaviside-Lorentz units.

Solution: The H-L value of the electron's charge is $\hat{e} = e(\text{in Coulombs})/\sqrt{\epsilon_0}$. This is $\approx 1.602 \times 10^{-19} / 2.976 \times 10^{-6} \approx 5.383 \times 10^{-14}$.

c) The combination $\hbar c$ also has dimensions of $(\text{Energy}) \times (\text{Length})$. From part b) determine the numerical value of the (dimensionless) fine structure constant $\alpha = \hat{e}^2 / 4\pi\hbar c$.

Solution: The SI value of $\hbar c \approx 3.162 \times 10^{-26}$. Then $\hat{e}^2 / 4\pi\hbar c \approx 0.7292 \times 10^{-2} \approx 1/137$.