Electromagnetic Theory I

Solution Set 2

Due: 16 September 2020

5. a) Prove that an electron placed at the center of a spherical cavity in a conductor is shielded from external gravitational fields. In other words the total force on that electron is zero when the conductor is held fixed in an arbitrary (static and Newtonian) gravitational field.

   Solution: Since the mobile charge carriers are electrons, static equilibrium is achieved when the total force on each electron, in the body of the conductor, $\boldsymbol{F} = -e\boldsymbol{E} + m_e\boldsymbol{g} = -\nabla(-e\phi_e + m_e\phi_g) = 0$. Thus $(-e\phi_e + m_e\phi_g) = C$, a constant throughout the body of the conductor. By uniqueness of solutions of Laplace's equation, $(-e\phi_e + m_e\phi_g) = C$ also in empty hollow cavities within the conductor. Since we place an electron at the center of a spherical cavity, the further charge rearrangement in the conductor due to its field will be uniformly distributed on the spherical wall and will exert no net force. Thus the electron will feel the force $-e\boldsymbol{E} + m_e\boldsymbol{g} = 0$.

   b) Prove that the instantaneous acceleration of a similarly placed positron (with charge equal and opposite to that of the electron and mass equal to that of the electron) is twice the normal gravitational acceleration.

   Solution: A positron has charge $+e$ so the force on it due to the external field will be $+e\boldsymbol{E} + m_e\boldsymbol{g} = 2m_e\boldsymbol{g}$ so its instantaneous acceleration is $2\boldsymbol{g}$.

   c) Find the instantaneous acceleration of a similarly placed muon, which has the same charge as the electron but a different mass $m_\mu > m_e$. (In fact the muon is about 200 times heavier than the electron).

   Solution: A muon has charge $-e$ so the force on it due to the external field will be $-e\boldsymbol{E} + m_\mu\boldsymbol{g} = (m_\mu - m_e)\boldsymbol{g}$ so its instantaneous acceleration is $(1 - m_e/m_\mu)\boldsymbol{g}$.

6. Four identical spherical conductors are placed with their centers at the corners of a square. Number them 1,2,3,4 as you go clockwise around the square. Consider the capacitor coefficients C_{ij} for this system.

   a) Using the symmetry of the problem, together with the reciprocity theorem, determine the number of independent coefficients C_{ij} and choose one of each.

   Solution: The symmetry of the square under a 90° rotation implies that $C_{11} = C_{22} = C_{33} = C_{44}$; $C_{12} = C_{23} = C_{34} = C_{41}$; and $C_{13} = C_{24}$. And of course $C_{ij} = C_{ji}$ by reciprocity. Thus there are only 3 independent coefficients, which we choose to be C_{11}, C_{12}, C_{13}.

b) A charge Q is placed on conductor 1, with 2,3,4 initially neutral. Then a wire connects and disconnects conductor 1 in turn with each of the other conductors in the order 2, 3, 4. Find the final charges Q_1, Q_2, Q_3, Q_4 on the four spheres in terms of Q and the independent C_{ij} you chose in part a).

Solution: When two conductors are connected by a wire, charge rearranges in such a way that both conductors are at the same potential. When 1 and 2 are connected the charge Q is shared equally between the two, because of symmetry. Thus $Q_2 = Q/2$. Now $Q/2$ resides on 1. Similarly when the 1 is connected to 3 symmetry dictates that $Q_3 = Q/4$, with $Q/4$ remaining on 1. Finally when 1 is connected to 4 we no longer have symmetry so we have to use the capacitor equations with $V_4 = V_1$:

$$\begin{aligned}
Q_1 &= C_{11}V_1 + C_{12}V_2 + C_{13}V_3 + C_{12}V_1 \\
Q_2 &= C_{12}V_1 + C_{11}V_2 + C_{12}V_3 + C_{13}V_1 = \frac{Q}{2} \\
Q_3 &= C_{13}V_1 + C_{12}V_2 + C_{11}V_3 + C_{12}V_1 = \frac{Q}{4} \\
Q_4 &= C_{12}V_1 + C_{13}V_2 + C_{12}V_3 + C_{11}V_1 = \frac{Q}{4} - Q_1
\end{aligned}$$

This gives 4 equations to determine the four unknowns V_1, V_2, V_3, Q_1. Taking the difference of the 2nd and 3rd equations as well as the difference of the 1st an 4th equations leads respectively to

$$(C_{11} - C_{12})(V_2 - V_3) = \frac{Q}{4}, \qquad (C_{12} - C_{13})(V_2 - V_3) = 2Q_1 - \frac{Q}{4} \tag{1}$$

which determines

$$Q_1 = \frac{Q}{8}\left(\frac{C_{11} - C_{13}}{C_{11} - C_{12}}\right), \qquad Q_2 = \frac{Q}{2}, \qquad Q_3 = \frac{Q}{4}, \qquad Q_4 = \frac{Q}{8}\left(\frac{C_{11} + C_{13} - 2C_{12}}{C_{11} - C_{12}}\right) \tag{2}$$

c) Repeat step b) but in the order 2, 4, 3.

Solution: The first step results in $Q_2 = Q/2$ as before. But now no symmetry applies in the second step so for this one we use the capacitor equations

$$\begin{aligned}
Q_1 &= C_{11}V_1 + C_{12}V_2 + C_{13}V_3 + C_{12}V_1 \\
Q_2 &= C_{12}V_1 + C_{11}V_2 + C_{12}V_3 + C_{13}V_1 = \frac{Q}{2} \\
Q_3 &= C_{13}V_1 + C_{12}V_2 + C_{11}V_3 + C_{12}V_1 = 0 \\
Q_4 &= C_{12}V_1 + C_{13}V_2 + C_{12}V_3 + C_{11}V_1 = \frac{Q}{2} - Q_1
\end{aligned}$$

The solution is similar to the last step of part a):

$$Q_1 = \frac{Q}{4}\left(\frac{C_{11} - C_{13}}{C_{11} - C_{12}}\right), \qquad Q_2 = \frac{Q}{2}, \qquad Q_3 = 0, \qquad Q_4 = \frac{Q}{4}\left(\frac{C_{11} + C_{13} - 2C_{12}}{C_{11} - C_{12}}\right)$$

For the last step we have symmetry about the diagonal from 2 to 4, so the remaining charge on 1 is equally divided between 1 and 3:

$$Q_1 = \frac{Q}{8}\left(\frac{C_{11} - C_{13}}{C_{11} - C_{12}}\right), \quad Q_2 = \frac{Q}{2}, \quad Q_3 = \frac{Q}{8}\left(\frac{C_{11} - C_{13}}{C_{11} - C_{12}}\right), \quad Q_4 = \frac{Q}{4}\left(\frac{C_{11} + C_{13} - 2C_{12}}{C_{11} - C_{12}}\right)$$

d) Compare and contrast the results of parts b) and c).

Solution: The order of connections matters. The final result in part b) has no symmetry: each charge is different. This contrasts with the final result for c) which has $Q_1 = Q_3$, a symmetry between 1 and 3. Although the answer to b) has less symmetry than c), it does have the feature that two of the charges are simple fractions of Q, whereas only one has that feature in c).

7. Using the Green function to solve an elementary problem can be a little like cracking a peanut with a sledgehammer. However, to test your understanding, apply this method to find the potential everywhere outside a spherical conductor of radius R which is held at the constant potential V.

a) First find the answer for the desired potential by elementary means (e.g. the method of images).

Solution: Put the image charge at the center of the sphere such that the potential at $r = R$ is V: $\phi = VR/r$.

b) The Dirichlet Green function for this problem is just proportional to the potential for a point charge outside the spherical conductor held at zero potential. Normalizing it appropriately, plug it into the explicit formula for the boundary value solution discussed in class (Lecture notes section 2.4 or Jackson (1.44)), and do the surface integral recovering the result found in part a). (N.B. Jackson's Green functions are a factor of 4π times my Green functions!)

Solution: The normal derivative of G_D for the sphere is

$$\hat{n} \cdot \nabla G_D = -\left.\frac{\partial G_D}{\partial r}\right|_{r=R} = \frac{R - r'^2/R}{4\pi(R^2 + r'^2 - 2Rr'\cos\theta)^{3/2}} \tag{3}$$

Then

$$\begin{aligned}
\phi(\boldsymbol{r}') &= -V \int d\Omega \hat{n} \cdot \nabla G_D(\boldsymbol{r}, \boldsymbol{r}') \\
&= -\frac{V}{2} R(R^2 - r'^2) \int_{-1}^{1} dz [R^2 + r'^2 - 2Rr'z]^{-3/2} \\
&= \frac{VR(r'^2 - R^2)}{2rr'} \left[\frac{1}{r' - R} - \frac{1}{r' + R}\right] = \frac{VR}{r'}
\end{aligned} \tag{4}$$

8. J, Problem 2.1

a) **Solution**: Choose the xy-plane to coincide with the conducting plane, and place q on the z-axis at $z = d$. Then the potential is given by

$$\phi(x, y, z) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}}\right) \tag{5}$$

The surface charge density is then

$$\sigma = \epsilon_0 E_z(z = 0) = -\epsilon_0 \frac{\partial \phi}{\partial z}\bigg|_{z=0} = -\frac{qd}{2\pi(x^2 + y^2 + d^2)^{3/2}} \tag{6}$$

b) **Solution**: The force on q is, by Coulomb's law,

$$\boldsymbol{F} = -\hat{z}\frac{q^2}{4\pi\epsilon_0(2d)^2} = -\hat{z}\frac{q^2}{16\pi\epsilon_0 d^2} \tag{7}$$

c) **Solution**: Doing the integral

$$F = \int dxdy \frac{\sigma^2}{2\epsilon_0} = 2\pi \int_0^\infty \rho d\rho \frac{q^2 d^2}{8\pi^2\epsilon_0(\rho^2 + d^2)^3} = \frac{q^2 d^2}{8\pi\epsilon_0}\int_0^\infty du \frac{1}{(u + d^2)^3} = \frac{q^2}{16\pi\epsilon_0 d^2} \tag{8}$$

d) **Solution**: The work required to move the charge from $z = d$ to $z = \infty$ is

$$W = \int_d^\infty dz \frac{q^2}{16\pi\epsilon_0 z^2} = \frac{q^2}{16\pi\epsilon_0 d} \tag{9}$$

e) **Solution**: $U = -\frac{q^2}{4\pi\epsilon_0(2d)} = -\frac{q^2}{8\pi\epsilon_0 d}$ This is twice in magnitude the result of part d). The reason is that part d) does not include the work done to move the image charge to infinity, which would be the same as that for the real charge. The system is not isolated since the conducting plane is connected to a battery in order that its potential stay zero as q is moved out to infinity.

f) **Solution**: We write the answer to part d) as (with $q_e = -e$)

$$W = \frac{e^2}{16\pi\epsilon_0\hbar c}\frac{\hbar c}{d} = \frac{\alpha}{4}m_e c^2 \frac{\hbar}{m_e c d} = \frac{1}{4}m_e c^2 \alpha^2 \frac{a_0}{d} \approx \frac{1}{2}(13.6 \text{ eV})\frac{0.5\text{Å}}{d} \to 3.4 \text{ eV} \tag{10}$$

for $d = 1$Å, where $\alpha = e^2/4\pi\epsilon_0\hbar c$ is the fine structure constant, and we have used the fact that the binding energy of hydrogen is $m_e c^2\alpha^2/2 \approx 13.6$ eV and the Bohr radius $a_0 = \hbar/m_e c\alpha \approx 0.5$Å