Electromagnetic Theory I

Solution Set 3

Due: 23 September 2020

9. The formula for the total energy stored in a static electric field $U = (\epsilon_0/2)\int d^3x \boldsymbol{E}^2$ includes the self-energy of the charges as well as their mutual potential energy. As discussed in class the self-energy of a point charge is infinite. Study this issue for a single charge Q by spreading out a point charge in two different ways: a) Replacing the point charge by a uniformly charged spherical surface of radius δ and b) Replacing the point charge by a uniformly charged ball of radius δ. In each case, find the electric field everywhere and evaluate the integral defining U. Classical physics puts no apriori limit on how small δ could be. However, relativistic quantum mechanics implies that a particle's location cannot be known within a distance smaller than its Compton wavelength $\hbar/mc$. For cases a) and b), assume $\delta = \hbar/mc$ to get an estimate for the self energy of the electron.

Solution: For $r > \delta$, the electric field is $\boldsymbol{E}_> = Q\hat{r}/4\pi\epsilon_0 r^2$ in both cases a) and b). So for this region we have

$$U_> = \frac{\epsilon_0}{2}\int_{r\geq\delta} d^3x \frac{Q^2}{16\pi^2\epsilon_0^2 r^4} = 4\pi\int_\delta^\infty dr \frac{Q^2}{32\pi^2\epsilon_0 r^2} = \frac{Q^2}{8\pi\epsilon_0}\int_\delta^\infty \frac{dr}{r^2} = \frac{Q^2}{8\pi\epsilon_0\delta} \tag{1}$$

a) In this case, $\boldsymbol{E} = 0$ for $r < \delta$, so the self energy for this model is

$$U_a = U_> = \frac{Q^2}{8\pi\epsilon_0\delta} \tag{2}$$

For the electron $Q = -e$, so setting $\delta = \hbar/m_e c$ we get the estimate $E_{\text{self}} \approx e^2 m_e c/8\pi\epsilon_0\hbar = m_e c^2\alpha/2$.

b) In this case, the electric field for $r < \delta$ is $\boldsymbol{E}_< = Q\mathbf{r}/(4\pi\epsilon_0\delta^3)$, and the energy for this region is

$$U_< = \frac{\epsilon_0}{2}\int_{r\leq\delta} d^3x \frac{Q^2 r^2}{16\pi^2\epsilon_0^2\delta^6} = 4\pi\int_\delta^\infty dr \frac{Q^2}{32\pi^2\epsilon_0\delta^6} = \frac{Q^2}{8\pi\epsilon_0}\int_0^\delta \frac{r^4 dr}{\delta^6} = \frac{Q^2}{40\pi\epsilon_0\delta} \tag{3}$$

Then

$$U_b = U_> + U_< = \frac{6Q^2}{40\pi\epsilon_0\delta} = \frac{3Q^2}{20\pi\epsilon_0\delta} \tag{4}$$

For the electron this model gives the self energy estimate $E_{\text{self}} \approx 3e^2 m_e c/20\pi\epsilon_0\hbar = 3m_e c^2\alpha/5$. We see that the estimate for the self-energy is model dependent, with model b) giving an estimate 20% higher than model a).

10. The standard capacitor used in electrical circuits has only two conducting plates and is usually assumed to remain neutral when charged by a battery. The charge-voltage relation is then simply $Q = CV$, where Q is the charge on one of the plates (so $-Q$ is the charge on the other plate) and V is the potential difference between the plates. C is called the capacitance of the capacitor. By specializing the general two conductor situation, described by $Q_i = \sum_{j=1,2} C_{ij}V_j$, to this case, derive an expression for C in terms of the C_{ij}.

Solution: Setting $Q_2 = -Q_1 = -Q$ the capacitor equations read

$$Q = C_{11}V_1 + C_{12}V_2, \qquad -Q = C_{21}V_1 + C_{22}V_2 \tag{5}$$

$$0 = (C_{11} + C_{21})V_1 + (C_{12} + C_{22})V_2, \qquad V_2 = -\frac{C_{11} + C_{21}}{C_{12} + C_{22}}V_1 \tag{6}$$

$$V = V_2 - V_1 = -\left(1 + \frac{C_{11} + C_{21}}{C_{12} + C_{22}}\right)V_1 = -\frac{C_{11} + C_{21} + C_{12} + C_{22}}{C_{12} + C_{22}}V_1 \tag{7}$$

$$Q = \left(\frac{C_{11}C_{22} - C_{12}C_{21}}{C_{12} + C_{22}}\right)V_1 = \frac{C_{11}C_{22} - C_{12}C_{21}}{C_{11} + C_{21} + C_{12} + C_{22}}V \equiv CV \tag{8}$$

$$C = \frac{C_{11}C_{22} - C_{12}C_{21}}{C_{11} + C_{21} + C_{12} + C_{22}} = \frac{C_{11}C_{22} - C_{12}^2}{C_{11} + 2C_{12} + C_{22}} \tag{9}$$

where the last equality used the reciprocity relation $C_{21} = C_{12}$.

11. Consider two spherical conductors of radius R, with their centers separated by a distance $D > 2R$. The first is grounded at 0 potential, and the second is held at potential V. The problem is to find the potential everywhere outside the two spheres. One can use the method of images, but an infinite sequence of image charges inside each sphere will be required.

a) Determine the location and size of these image charges by relating the location and size of the nth charge to those of the $n-1$th charge. (Hint: Start with a charge at the center of the second conductor, with a size that brings the surface of that conductor to the desired potential V. The next image charge will be inside the first conductor of strength and location chosen so that the first conductor has $\phi = 0$, and so on.)

Solution: The determination of image charges is recursive. Starting with $q_0 = 4\pi\epsilon_0 RV$ at the center of sphere II, we find its image in sphere I, the image of that in II, and so on. It is convenient to call r_k the distance from the center of the sphere q_k resides in. Then $r_0 = 0$, $r_1 = R^2/D$, $\cdots, r_k = R^2/(D - r_{k-1})$. Then $q_k = -q_{k-1}R/(D - r_{k-1}) = -q_{k-1}r_k/R$. Setting the origin of coordinates at the midpoint of the line connecting the centers of the spheres, we have for the coordinate locations:

$$x_k = \frac{D}{2} - r_k, \qquad \text{for } k \text{ odd}, \qquad x_k = -\frac{D}{2} + r_k, \qquad \text{for } k \text{ even} \tag{10}$$

The charges in sphere I are negative and those in II are positive.

b) The sequence of charges inside each conductor approaches a limiting location–find the limiting location in both conductors.

Solution: The limiting radius as $k \to \infty$ satisfies $r_\infty = R^2/(D - r_\infty)$ which is a quadratic equation with roots $r_\pm = \frac{D}{2} \pm \sqrt{\frac{D^2}{4} - R^2}$. Since $r_= r_- = R^2$, $r_- < R$ is the one we identify as r_∞. Inside sphere I, $x^I_\infty = -D/2 + r_- = -\sqrt{D^2 - 4R^2}/2$. Inside II $x^{II}_\infty = D/2 - r_- = \sqrt{D^2 - 4R^2}/2$

c)Show that the infinite sum of potentials from the image charges converges. By keeping enough of the terms for the case $D = 4R$, compute the potential as a multiple of V, at the point midway between the two conductors on the line joining their centers, to 3 significant figures.

Solution: For k sufficiently large $q_k \sim (-r_\infty q_{k-1}/R)$, so successive terms decay exponentially in k, since $r_\infty/R < 1$.
The total potential at $x = 0$ is

$$V = \sum_{k=0}^{\infty} \frac{q_k}{4\pi\epsilon_0 |x_k|} = \sum_{k=0}^{\infty} \frac{q_k}{4\pi\epsilon_0 |D/2 - r_k|} \tag{11}$$

To evaluate this numerically for $D = 4R$, set $r_k = \rho_k R$ and $q_k = 4\pi\epsilon_0 V R e_k$ so that

$$\begin{aligned} \phi(0) &= V \sum_{k=0}^{\infty} \frac{e_k}{2 - \rho_k} \\ \rho_k &= \frac{1}{4 - \rho_{k-1}}, \qquad e_k = -e_{k-1}\rho_k, \qquad \rho_0 = 0, \quad e_0 = 1 \end{aligned} \tag{12}$$

$\rho_1 = 1/4, \rho_2 = 4/15, \rho_3 = 15/56,\ e_1 = -1/4, e_2 = 1/15, e_3 = -1/56,$

$$\phi(0) = V\left[\frac{1}{2} - \frac{1}{7} + \frac{1}{26} - \frac{1}{97} + \cdots\right] \approx 0.39V \tag{13}$$

d) Calculate the total charge on each conductor to 3 significant figures as a multiple of $4\pi\epsilon_0 V R$ again in the case $D = 4R$.

Solution: The charges on each sphere are

$$Q_{\rm I} = 4\pi\epsilon_0 V R \sum_{k=odd} e_k, \qquad Q_{\rm II} = 4\pi\epsilon_0 V R \sum_{k=even} e_k \tag{14}$$

For 3 figure accuracy we need a few more k values than given in c): For $k > 3$, $\rho_k \approx \rho_\infty = 2 - \sqrt{3} \approx 0.2679$ to the necessary accuracy.

$$\begin{aligned} \frac{Q_{\rm I}}{4\pi\epsilon_0 V R} &\approx e_1 + e_3 + e_5 \approx -0.269, \\ \frac{Q_{\rm II}}{4\pi\epsilon_0 V R} &\approx e_0 + e_2 + e_4 \approx 1 + 0.0667 + .0048 \approx 1.072 \end{aligned}$$

12. J, problem 2.23.

a) To make the potentials on the faces perpendicular to the z-axis the same, choose the $Z(z) \propto \cosh\gamma(z-a/2)$. Then the series for the potential is

$$\phi = \sum_{m,n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{m\pi y}{a} \cosh\frac{(2z-a)\pi}{2a}\sqrt{m^2+n^2} \tag{15}$$

Then $\phi(z=0,a)=V$ and orthogonality of the sin functions leads to

$$\begin{aligned} A_{mn} &= \frac{16V}{mn\pi^2\cosh(\pi\sqrt{m^2+n^2}/2)}\delta_{m,odd}\delta_{n,odd} \\ \phi &= \frac{16V}{\pi^2}\sum_{m,n=odd}\frac{\cosh{(2z-a)\pi\sqrt{m^2+n^2}/2a}}{mn\cosh(\pi\sqrt{m^2+n^2}/2)}\sin\frac{m\pi x}{a}\sin\frac{m\pi y}{a} \end{aligned} \tag{16}$$

b) For $m=2k+1$, $\sin m\pi/2 = (-)^k$, so

$$\phi_{\text{center}} = \frac{16V}{\pi^2}\sum_{k,l=0}^{\infty}\frac{(-)^{k+l}}{(2k+1)(2l+1)\cosh\pi\sqrt{(2k+1)^2+(2l+1)^2}/2} \tag{17}$$

which is rapidly convergent. For example the first term is $0.3475V$ which is not very far from $V/3$, the average of the potential on the walls of the box. The first 3 terms improves this to $0.332V$.

c)

$$\begin{aligned} \sigma(z=a) &= \epsilon_0(-\hat{z})\cdot\boldsymbol{E} = \epsilon_0\left.\frac{\partial\phi}{\partial z}\right|_{z=a} \\ &= \frac{16V\epsilon_0}{\pi a}\sum_{m,n=odd}\frac{\sqrt{m^2+n^2}}{mn}\tanh(\pi\sqrt{m^2+n^2}/2)\sin\frac{m\pi x}{a}\sin\frac{m\pi y}{a} \end{aligned} \tag{18}$$