Electromagnetic Theory I

Solution Set 4

Due: 30 September 2020

13. We have derived the spherical harmonic $Y_{ll}(\theta,\varphi)$,

$$Y_{ll}(\theta,\varphi) = \frac{(-)^l}{l!\sqrt{2\pi}}\sqrt{\frac{(2l+1)!}{2^{2l+1}}}\,\sin^l\theta e^{il\varphi}, \tag{1}$$

normalized so that $\int d\Omega |Y_{lm}|^2 = 1$, as a solution of the differential equation $L_+Y_{ll} = 0$ where $L_\pm = L_x \pm iL_y$ are the angular momentum ladder operators. Recall that $L_\pm$ raise or lower the m value of a spherical harmonic Y_{lm} by one unit:

$$L_\pm Y_{lm} = \sqrt{l(l+1)-m(m\pm1)}Y_{lm\pm1} = \sqrt{(l\mp m)(l\pm m+1)}Y_{lm\pm1}\ . \tag{2}$$

Using these facts, apply L_- repeatedly to Y_{ll} to derive all the Y_{lm} for $0\le m\le l$ for the cases a) $l=1$, b) $l=2$, and c) $l=3$. You may use without proof the spherical coordinate form of the $L_\pm, L_z$:

$$L_\pm = \pm e^{\pm i\varphi}\left(\frac{\partial}{\partial\theta}\pm i\cot\theta\frac{\partial}{\partial\varphi}\right), \qquad L_z = \frac{1}{i}\frac{\partial}{\partial\varphi} \tag{3}$$

Solution: Notice that since $\partial Y_{lm}/\partial\varphi = imY_{lm}$, L_- applied to Y_{lm} reduces to

$$L_-Y_{lm} = -e^{-i\varphi}\left(\frac{\partial}{\partial\theta}+m\cot\theta\right)Y_{lm} \tag{4}$$

Exploiting this simplification, we evaluate

a)

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi} \tag{5}$$

$$\begin{aligned} Y_{10} &= \frac{1}{\sqrt{2}}L_-Y_{11} \\ &= -\sqrt{\frac{3}{16\pi}}L_-\sin\theta e^{i\varphi} = \sqrt{\frac{3}{16\pi}}\left(\frac{\partial}{\partial\theta}+\cot\theta\right)\sin\theta = \sqrt{\frac{3}{4\pi}}\cos\theta \end{aligned} \tag{6}$$

b)

$$
\begin{aligned}
Y_{22} &= \sqrt{\frac{5!}{2^8\pi}}\,\sin^2\theta e^{2i\varphi} = \sqrt{\frac{15}{32\pi}}\,\sin^2\theta e^{2i\varphi} \\
Y_{21} &= \frac{1}{\sqrt{4}}L_-Y_{22} = -\sqrt{\frac{15}{128\pi}}e^{i\varphi}\left(\frac{\partial}{\partial\theta}+2\cot\theta\right)\sin^2\theta \\
&= -\sqrt{\frac{15}{8\pi}}e^{i\varphi}\sin\theta\cos\theta \\
Y_{20} &= \frac{1}{\sqrt{6}}L_-Y_{21} = \sqrt{\frac{5}{16\pi}}\left(\frac{\partial}{\partial\theta}+\cot\theta\right)\sin\theta\cos\theta \\
&= \sqrt{\frac{5}{16\pi}}\left(2\cos^2\theta-\sin^2\theta\right) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta-1\right) \qquad (7)
\end{aligned}
$$

c)

$$
\begin{aligned}
Y_{33} &= -\frac{1}{6}\sqrt{\frac{7!}{2^8\pi}}\,\sin^3\theta e^{3i\varphi} = -\sqrt{\frac{35}{64\pi}}\,\sin^3\theta e^{3i\varphi} \\
Y_{32} &= \frac{1}{\sqrt{6}}L_-Y_{33} = \sqrt{\frac{35}{384\pi}}\left(\frac{\partial}{\partial\theta}+3\cot\theta\right)\sin^3\theta e^{2i\varphi} \\
&= \sqrt{\frac{105}{32\pi}}\sin^2\theta\cos\theta e^{2i\varphi} \\
Y_{31} &= \frac{1}{\sqrt{10}}L_-Y_{32} = -\sqrt{\frac{105}{320\pi}}\left(\frac{\partial}{\partial\theta}+2\cot\theta\right)\sin^2\cos\theta e^{i\varphi} \\
&= -\sqrt{\frac{21}{64\pi}}\left(4\sin\theta\cos^2\theta-\sin^3\theta\right)e^{i\varphi} = -\sqrt{\frac{21}{64\pi}}\sin\theta\left(5\cos^2\theta-1\right)e^{i\varphi} \\
Y_{30} &= \frac{1}{\sqrt{12}}L_-Y_{31} = \sqrt{\frac{7}{256\pi}}\left(\frac{\partial}{\partial\theta}+\cot\theta\right)\sin\theta\left(5\cos^2\theta-1\right) \\
&= \sqrt{\frac{7}{256\pi}}\left(5\cos^3\theta-\cos\theta-10\sin^2\theta\cos\theta+5\cos^3\theta-\cos\theta\right) \\
&= \sqrt{\frac{7}{256\pi}}\left(20\cos^3\theta-12\cos\theta\right) = \sqrt{\frac{7}{16\pi}}\left(5\cos^3\theta-3\cos\theta\right) \qquad (8)
\end{aligned}
$$

14. One can sometimes use part of the solution of a simple potential problem as the solution of an apparently more complicated problem. For example, any equipotential surface of the simple problem can be replaced by a conductor of coincident shape.

a) First, find the potential everywhere outside an isolated conducting sphere of radius R carrying charge Q, immersed in a uniform external electric field. (Hint: Take the z-axis parallel to the external field, expressing its potential in spherical coordinates, taking the center of the sphere at the origin.)

Solution: With $\boldsymbol{E} = E\hat{z}$, the potential for this external field is $\phi = -Ez = -Er\cos\theta = -ErP_1(\cos\theta)$. Azimuthal symmetry implies that the potential for this system has the expansion

$$\phi = \sum_{l=0}^{\infty}(a_l r^l + b_l/r^{l+1})P_l(\cos\theta) \tag{9}$$

Matching this to the external field at large r implies $a_1 = -E$ and $a_l = 0$ for $l > 1$. $\phi = V$, a constant, on the sphere then implies $b_1 = ER^3$ and $b_l = 0$ for $l > 1$. Then

$$\phi = a_0 + \frac{b_0}{r} + E\left(\frac{R^3}{r^2} - r\right)\cos\theta \tag{10}$$

Here $b_0 = Q/4\pi\epsilon_0$ with Q the total charge on the sphere. For a neutral sphere $b_0 = 0$.

b) Now consider a hemispherical conductor of radius R attached to a grounded plane (the xy-plane) (i.e. the hemisphere-plane combination are all at zero potential). The top of the hemisphere is at $z = R$ and its center is at the origin of coordinates. The electric field, in the (empty) region above this conducting surface, approaches a uniform field $E_0\hat{z}$ far from the hemisphere. Calculate the surface charge density everywhere on the conducting surface (plane and hemisphere).

Solution: Note that the solution of part a), with $a_0 = b_0 = 0$, $E = E_0$, satisfies the boundary conditions of this problem if the conductor with the hemisphere is taken in the xy-plane with the center of the hemisphere at the origin of coordinates and the top of the boss at $z = R$. This is because $\phi = 0$ at $\theta = \pi/2$. The unique solution is therefore

$$\phi \;=\; E_0\left(\frac{R^3}{r^2} - r\right)\cos\theta. \tag{11}$$

The surface chage density is given by $\sigma = \epsilon_0 E_n$. On the hemisphere $E_n = -\partial\phi/\partial r = (1 + 2R^3/r^3)E_0\cos\theta \to 3E_0\cos\theta$ at $r = R$. On the plane outside the hemisphere $E_n = -\partial\phi/\partial z = (1 - R^3/r^3)E_0$ at $z = 0$.

c) Calculate the total charge on the hemisphere of part b), in terms of E_0 and R.

Solution: $Q_{\text{hemisph}} = 2\pi R^2 \int_0^{\pi/2} \sin\theta d\theta 3\epsilon_0 E_0 \cos\theta = 3\pi R^2\epsilon_0 E_0$.

15. J, Problem 3.4.

Solution: We can use the Green function in spherical coordinates for the interior of a sphere plus Green's theorem to give an expansion for the potential inside the sphere. We specialize the Green function for the space between two concentric shells to this case by taking the inner radius $a \to 0$:

$$\begin{aligned} G_D(\boldsymbol{r}, \boldsymbol{r}') &= \sum_{l=0}^{\infty} r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right) \frac{1}{2l+1} \sum_{m=-l}^{+l} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') \\ \hat{n} \cdot \nabla' G_D \bigg|_{r'=b} &= - \sum_{l=0}^{\infty} \frac{r^l}{b^{l+2}} \sum_{m=-l}^{+l} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') \end{aligned} \quad (12)$$

a) Green's theorem for this geometry reads

$$\phi(\boldsymbol{r}) = - \int b^2 d\Omega' \phi(\boldsymbol{r}') \hat{n} \cdot \nabla' G_D(\boldsymbol{r}, \boldsymbol{r}') = \sum_{l=0}^{\infty} \frac{r^l}{b^l} \sum_{m=-l}^{+l} Y_{lm}(\theta, \varphi) \int d\Omega' Y_{lm}^*(\theta', \varphi') V(\varphi)$$

For this problem the φ' integral is proportional to

$$\int_0^{2\pi} d\varphi' e^{-im\varphi} V(\varphi') = V \sum_{k=0}^{2n-1} (-)^k \int_{k\pi/n}^{(k+1)\pi/n} d\varphi' e^{-im\varphi'} = V \sum_{k=0}^{2n-1} \frac{(-)^k}{-im} e^{-i(k+1)m\pi/n} \left(1 - e^{im\pi/n}\right)$$

The sum over k is a geometric sum which is zero unless m/n is an odd integer. If it is an odd integer, the result is simply

$$\int_0^{2\pi} d\varphi' e^{-im\varphi} V(\varphi') = \frac{4nV}{im}, \qquad \frac{m}{n} = \text{odd} \quad (13)$$

Then the expansion coefficients can be written

$$\int d\Omega' Y_{lm}^*(\theta', \varphi') V(\varphi') = \delta_{m/n=\text{odd}} \frac{4nV}{im} \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} \int_{-1}^{+1} d\cos\theta P_l^m(\cos\theta)$$

Since $P_l^m(-x) = (-)^{l+m} P_l^m(x)$, we also see that the coefficient is zero unless $l+m$ is even.

b) For the case $n=1$ both m and l are restricted to be odd. so we need to evaluate the coefficients for $(lm) = (1, \pm 1), (3, \pm 3), (3, \pm 1)$. Furthermore, the coefficient for $-m$ is just $-$ the complex conjugate of the coefficient for m, so we just need the cases $(11), (33), (31)$.

$$\begin{aligned} \int d\Omega' Y_{11}^*(\theta', \varphi') V(\varphi') &= -4iV(-) \sqrt{\frac{3}{8\pi}} \int_0^{\pi} d\theta \sin^2\theta = 4iV \frac{\pi}{2} \sqrt{\frac{3}{8\pi}} \\ \int d\Omega' Y_{33}^*(\theta', \varphi') V(\varphi') &= \frac{4iV}{3} \sqrt{\frac{35}{64\pi}} \int_0^{\pi} d\theta \sin^4\theta = \frac{4iV}{3} \frac{3\pi}{8} \sqrt{\frac{35}{64\pi}} \\ \int d\Omega' Y_{31}^*(\theta', \varphi') V(\varphi') &= 4iV \sqrt{\frac{21}{64\pi}} \int_0^{\pi} d\theta \sin^2\theta (4 - 5\sin^2\theta) = 4iV \frac{\pi}{8} \sqrt{\frac{21}{64\pi}} \end{aligned} \quad (14)$$

Then

$$
\begin{aligned}
\phi &= \frac{r}{b}(Y_{11}+Y_{1,-1})4iV\frac{\pi}{2}\sqrt{\frac{3}{8\pi}} + \frac{r^3}{b^3}(Y_{33}+Y_{3,-3})\frac{4iV}{3}\frac{3\pi}{8}\sqrt{\frac{35}{64\pi}} \\
&\quad + \frac{r^3}{b^3}(Y_{31}+Y_{3,-1})4iV\frac{\pi}{8}\sqrt{\frac{21}{64\pi}} + \cdots \\
&= \frac{3V}{2}\frac{r}{b}\sin\theta\sin\varphi + \frac{35V}{64}\frac{r^3}{b^3}\sin^3\theta\sin 3\varphi + \frac{21V}{64}\frac{r^3}{b^3}\sin\theta(4-5\sin^2\theta)\sin\varphi + \cdots \\
&= \frac{3V}{2}\frac{r}{b}\sin\theta\sin\varphi - \frac{35V}{16}\frac{r^3}{b^3}\sin^3\theta\sin^3\varphi + \frac{21V}{16}\frac{r^3}{b^3}\sin\theta\sin\varphi + \cdots \\
&= \frac{3V}{2}\frac{r}{b}P_1(\sin\theta\sin\varphi) - \frac{7V}{8}\frac{r^3}{b^3}P_3(\sin\theta\sin\varphi) + \cdots \qquad (15)
\end{aligned}
$$

This agrees with J, (3.36) because $\sin\theta\sin\varphi$ is the cosine of the angle between the observation point and the y-axis in this problem, which plays the role of the z axis in (3.36).

16. J, Problem 3.14.

Solution: First normalize the linear charge density to total charge Q: $\lambda = 3Q(d^2 - z^2)/4d^3$.

a) Use the expansion of the Dirichlet Green function in Legendre polynomials:

$$
G_D(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{4\pi}\sum_{l=0}^{\infty} r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)P_l(\cos\gamma) \qquad (16)
$$

where γ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$. The potential is then

$$
\phi = \frac{1}{\epsilon_0}\int d^3r' G_D(\boldsymbol{r},\boldsymbol{r}')\rho(\boldsymbol{r}') = \frac{3Q}{4\epsilon_0 d^3}\int dz'(d^2 - z'^2)G_D(\boldsymbol{r}, z'\hat{z}) \qquad (17)
$$

for $z' > 0$, $r' = z'$ and $\gamma = \theta$, and for $z' < 0$, $r' = -z'$ and $\gamma = \pi - \theta$. then

$$
\begin{aligned}
\phi &= \frac{3Q}{16\pi\epsilon_0 d^3}\sum_{l=0}^{\infty}(P_l(\cos\theta) + P_l(-\cos\theta))\int_0^d dr'(d^2 - r'^2)r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \\
&= \frac{3Q}{8\pi\epsilon_0 d^3}\sum_{l=even} P_l(\cos\theta)\int_0^d dr'(d^2 - r'^2)r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)
\end{aligned}
$$

When $r > d$, $r' < r$ and the integral is

$$
\begin{aligned}
\int_0^d dr'(d^2 - r'^2)r'^l\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right) &= d^{l+3}\left(\frac{1}{l+1} - \frac{1}{l+3}\right)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right) \\
&= d^{l+3}\frac{2}{(l+1)(l+3)}\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)
\end{aligned}
$$

$$
\phi = \frac{Q}{4\pi\epsilon_0}\sum_{l=even}\frac{3}{(l+1)(l+3)}\left(\frac{d^l}{r^{l+1}} - \frac{r^l d^l}{b^{2l+1}}\right)P_l(\cos\theta), \qquad r > d \qquad (18)
$$

When $r < d$ the integrands for $0 < r' < r$ and $r < r' < d$ are different:

$$\begin{aligned}
\int_0^r dr'(d^2 - r'^2) r'^l \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right) &= r^{l+1}\left(\frac{d^2}{l+1} - \frac{r^2}{l+3}\right)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right) \\
\int_r^d dr'(d^2 - r'^2) r^l \left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right) &= r^l\left(\frac{d^2}{l}\frac{d^l - r^l}{d^l r^l} - \frac{d^2}{l+1}\frac{d^{l+1} - r^{l+1}}{b^{2l+1}}\right. \\
&\quad \left. - \frac{1}{l-2}\frac{d^{l-2} - r^{l-2}}{d^{l-2} r^{l-2}} + \frac{1}{l+3}\frac{d^{l+3} - r^{l+3}}{b^{2l+1}}\right) \\
\int_0^d dr'(d^2 - r'^2) r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) &= \frac{d^2}{l+1} - \frac{r^2}{l+3} + \frac{d^2}{l} - \frac{r^2}{l-2} \\
&\quad + r^l\left(-\frac{d^2}{l d^l} - \frac{d^{l+3}}{(l+1) b^{2l+1}} + \frac{d^2}{(l-2) d^l} + \frac{d^{l+3}}{(l+3) b^{2l+1}}\right) \\
&= \frac{(2l+1) d^2}{l(l+1)} - \frac{(2l+1) r^2}{(l+3)(l-2)} \\
&\quad + r^l\left(\frac{2d^2}{l(l-2) d^l} - \frac{2d^{l+3}}{(l+1)(l+3) b^{2l+1}}\right)
\end{aligned}$$

$$\begin{aligned}
\phi &= \frac{Q}{4\pi\epsilon_0}\sum_{l=even}\left(\frac{3r^l}{l(l-2) d^{l+1}} - \frac{3 d^l r^l}{(l+1)(l+3) b^{2l+1}}\right) P_l(\cos\theta) \\
&\quad + \frac{Q}{4\pi\epsilon_0}\sum_{l=even}\frac{3}{2}\left(\frac{(2l+1)}{l(l+1) d} - \frac{(2l+1) r^2}{(l+3)(l-2) d^3}\right) P_l(\cos\theta), \qquad r < d
\end{aligned}$$

Here the terms on the first line satisfy Laplace's equation, but those on the second line satisfy Poisson's equation with the line charge density. The terms with $l = 0, 2$ have $0/0$ pieces that should be separated and handled carefully:

$$\begin{aligned}
l = 0: &\qquad \frac{3Q}{8\pi\epsilon_0}\frac{(2l+1)/(l+1) - (r/d)^l}{l d} \rightarrow \frac{3Q}{8\pi\epsilon_0}\frac{1}{d}\left(1 - \ln\frac{r}{d}\right) \\
l = 2: &\qquad \frac{3Q}{8\pi\epsilon_0}\frac{r^2}{d^3}\frac{(r/d)^{l-2} - (2l+1)/(l+3)}{l-2} \rightarrow \frac{3Q}{8\pi\epsilon_0}\frac{r^2}{d^3}\left(\ln\frac{r}{d} - \frac{1}{5}\right)
\end{aligned} \tag{19}$$

b) To get the surface charge density we need the normal component of the electric field $\boldsymbol{E} = -\nabla\phi$ on the bounding surface $r = b > d$: $\sigma = \epsilon_0 \hat{\boldsymbol{n}} \cdot \boldsymbol{E}$, where $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{r}}$ is directed away from the conductor. Thus

$$\begin{aligned}
\sigma &= \epsilon_0\frac{\partial\phi}{\partial r} = \frac{Q}{4\pi}\sum_{l=even}\frac{3}{(l+1)(l+3)}\left(-(l+1)\frac{d^l}{b^{l+2}} - l\frac{d^l}{b^{l+2}}\right) P_l(\cos\theta) \\
&= -\frac{Q}{4\pi b^2}\sum_{l=even}\frac{3(2l+1)}{(l+1)(l+3)}\frac{d^l}{b^l} P_l(\cos\theta)
\end{aligned} \tag{20}$$

c) When $d \ll b$, $\sigma \approx -Q/4\pi b^2$, so that the induced charge is uniformly distributed on the bounding surface. This is reasonable since the surface just sees a point charge at the sphere's center. For most of the interior of the sphere, $r \gg d$ and so $\phi \approx Q/4\pi\epsilon_0 r$. However, for $r \sim d \ll b$ the potential looks like a line charge in empty space:

$$\begin{aligned}
\phi &\approx \frac{Q}{4\pi\epsilon_0}\sum_{l=even}\frac{3}{(l+1)(l+3)}\frac{d^l}{r^{l+1}}P_l(\cos\theta), \qquad r > d \\
\phi &= \frac{Q}{4\pi\epsilon_0}\sum_{l=even}\frac{3r^l}{l(l-2)d^{l+1}}P_l(\cos\theta) \\
&\quad + \frac{Q}{4\pi\epsilon_0}\sum_{l=even}\frac{3}{2}\left(\frac{(2l+1)}{l(l+1)d} - \frac{(2l+1)r^2}{(l+3)(l-2)d^3}\right)P_l(\cos\theta), \qquad r < d \qquad (21)
\end{aligned}$$

where again the $l = 0, 2$ terms must be handled as in the general b case.