# Electromagnetic Theory I

Solution Set 5

Due: 7 October 2020

17. Following the method discussed in class for the Dirichlet Green function, obtain the mixed Dirichlet/Neumann Green function for the region between two concentric spheres of radii $a < b$ with Neumann conditions ($\partial G/\partial r = 0$) on the inner shell and Dirichlet conditions $G = 0$ on the outer shell.

**Solution**:

a) For this case the combination $(l+1)r^l + la^{2l+1}/r^{l+1}$ has vanishing radial derivative at $r = a$. (For $l = 0$, note that this expression is just a constant!) Thus we start with the empty space expansion and make the substitution:

$$\begin{aligned}
\frac{r_<^l}{r_>^{l+1}} &\to C\left(r_<^l + \frac{la^{2l+1}}{(l+1)r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \\
&= C\left(\frac{r_<^l}{r_>^{l+1}} - \frac{(rr')^l}{b^{2l+1}} + \frac{la^{2l+1}}{(l+1)(rr')^{l+1}} - \frac{la^{2l+1}r_>^l}{(l+1)b^{2l+1}r_<^{l+1}}\right) \\
&= C\Bigg(\frac{r_<^l}{r_>^{l+1}} - \frac{(rr')^l}{b^{2l+1}} + \frac{la^{2l+1}}{(l+1)(rr')^{l+1}} - \frac{la^{2l+1}r^l}{(l+1)b^{2l+1}r'^{(l+1)}} - \frac{la^{2l+1}r'^l}{(l+1)b^{2l+1}r^{l+1}} \\
&\quad + \frac{la^{2l+1}r_<^l}{(l+1)b^{2l+1}r_>^{l+1}}\Bigg)
\end{aligned}$$

To produce the correct delta function on the right side of the Green function equation we simply match coefficients of $r_<^l/r_>^{l+1}$: $C = (1 + la^{2l+1}/(l+1)b^{2l+1})^{-1}$. Thus the appropriate Green function is:

$$G_{ND}(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{4\pi}\sum_{l=0}^{\infty} \frac{\left(r_<^l + la^{2l+1}/(l+1)r_<^{l+1}\right)\left(1/r_>^{l+1} - r_>^l/b^{2l+1}\right)}{1 + la^{2l+1}/(l+1)b^{2l+1}} P_l(\cos\gamma)$$

Where $\gamma$ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$, $\boldsymbol{r}\cdot\boldsymbol{r}' = rr'\cos\gamma$. The $1/4\pi$ arranges the normalization in our lecture notes.

18. Use the Green function found in the previous problem to calculate

a) the potential and

**Solution**: The potential satisfies $\phi = V_0\cos\theta = V_0\sqrt{4\pi/3}Y_{10}$ at $r = b$ and $\partial\phi/\partial r = 0$ at $r = a$. Thus we use the first Green function of the previous problem in Green's theorem. For this we need the normal derivative of $G_{ND}$ at $r = b$:

$$\hat{n}\cdot\nabla' G_{ND}\Big|_{r'=b} = \sum_{l=0}^{\infty} \frac{\left(r^l + la^{2l+1}/(l+1)r^{l+1}\right)\left(-1/b^{l+2}\right)}{1 + la^{2l+1}/(l+1)b^{2l+1}} \sum_{m=-l}^{+l} Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')$$

Then since the spherical harmonics are orthonormal we have only the $l = 1, m = 0$ term contributing:

$$\begin{aligned}
\phi &= -\oint b^2 d\Omega' \frac{\partial G_{ND}}{\partial r'}\bigg|_{r=b} V_0 \sqrt{\frac{4\pi}{3}} Y_{10}(\Omega') \\
&= b^2 \frac{(r + a^3/2r^2)(1/b^3)}{1 + a^3/2b^3} V_0 \sqrt{\frac{4\pi}{3}} Y_{10}(\Omega) = \frac{(r^3 + a^3/2)}{b^3 + a^3/2} \frac{b^2 V_0}{r^2} \cos\theta
\end{aligned}$$

b) The electric field components

$$E_r = -\frac{\partial\phi}{\partial r} = -\frac{V_0 b^2 \cos\theta}{b^3 + a^3/2}\left[1 - \frac{a^3}{r^3}\right], \qquad E_\theta = -\frac{\partial\phi}{r\partial\theta} = \frac{V_0 b^2 \sin\theta}{b^3 + a^3/2}\left[1 + \frac{a^3}{2r^3}\right]$$

The cylindrical components of $\boldsymbol{E}$ are

$$\begin{aligned}
E_\rho &= E_r \sin\theta + E_\theta \cos\theta = \frac{3E_0 a^3 b^3 \sin\theta\cos\theta}{2(b^3 - a^3)r^3} = \frac{3E_0 a^3 b^3 \rho z}{2(b^3 - a^3)(\rho^2 + z^2)^{5/2}} \\
E_z &= E_r \cos\theta - E_\theta \sin\theta = -\frac{E_0 b^3}{b^3 - a^3}\left[1 - \frac{a^3}{2r^3}(3\cos^2\theta - 1)\right] \\
&= -\frac{E_0 b^3}{b^3 - a^3}\left[1 - \frac{a^3}{2}\frac{2z^2 - \rho^2}{(\rho^2 + z^2)^{5/2}}\right]
\end{aligned}$$

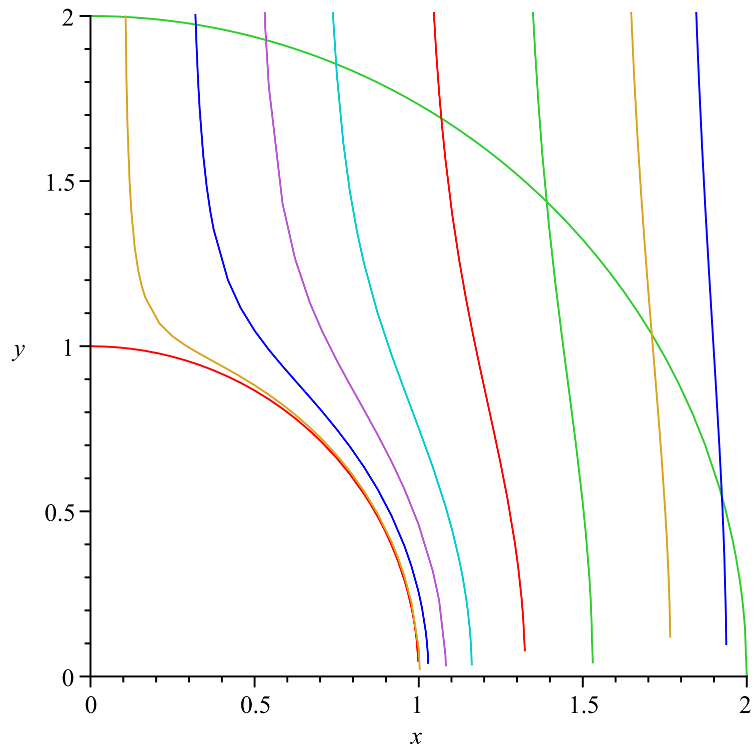
in the region $a < r < b$ for the situation in which the normal component of the electric field, $E_r$, vanishes on the inner sphere, and the potential on the outer sphere has the behavior $\phi(b, \theta) = V_0 \cos\theta$.

c) Sketch the electric field lines in the case where $a/b = 0.5$. The electric field lines are those curves that are everywhere tangent to the electric field.

**Solution**: Each field line has its tangent parallel to the field, i.e.

$$\begin{aligned}
\frac{1}{r}\frac{dr}{d\theta} &= \frac{E_r}{E_\theta} = -\cot\theta \frac{r^3 - a^3}{r^3 + a^3/2}, \qquad \sin\theta = \frac{Ka\sqrt{r}}{\sqrt{r^3 - a^3}} \\
\rho &= r\sin\theta = Ka\sqrt{\frac{r^3}{r^3 - a^3}}, \qquad z = r\cos\theta = \sqrt{a^2\left(\frac{\rho^2}{\rho^2 - K^2 a^2}\right)^{2/3} - \rho^2}
\end{aligned}$$

The differential equation for a field line $r(\theta)$ is solved by quadratures. Plotting $y = z/a$ versus $x = \rho/a$ for $K = 0.1, 0.3, 0, 5, 0.7, 1, 1.3, 1.6, 1.8$ gives:




19. The general electrostatic problem to find the potential in the interior of a hollow right circular cylinder of radius $a$ and length $L$, given its value on the boundary, was partially solved in class by finding the solution with an arbitrary potential on the face of one end, but vanishing potential at the other end and on the cylindrical surface.

a) Complete the solution, using cylindrical coordinates, by finding the potential inside the cylinder which vanishes on the two end faces, but is an arbitrary function $V(\phi, z)$ on the cylindrical boundary surface. Note that for these boundary conditions, the separation of variables constants should be chosen so $Z$ involves trigonometric functions. The solution will then involve a double Fourier series.

Solution: Since $\phi = 0$ at $z = 0, L$ it is appropriate to use $\sin(n\pi z/L)$ for the $z$ dependence:

$$\phi = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{im\varphi} \sin \frac{n\pi z}{L} I_m(n\pi\rho/L)$$

We do not include $K_m$ in the possible $\rho$ dependence since we need good behavior at $\rho = 0$. The boundary condition $\phi(a, \varphi, z) = V(\varphi, z)$ yields using orthogonality

$$A_{mn} I_m(n\pi a/L) = \frac{1}{\pi L} \int_0^{2\pi} d\varphi \int_0^L dz e^{-im\varphi} \sin \frac{n\pi z}{L} V(z, \varphi)$$

b) Evaluate the expansion coefficients for the case where $V(\phi, z) = +V$ for $0 \le \phi \le \pi$ and $V(\phi, z) = 0$ for $\pi < \phi < 2\pi$.

Solution: for $m \neq 0$,

$$\begin{aligned} A_{mn} I_m(n\pi a/L) &= \frac{V}{\pi L}\int_0^{\pi} d\varphi \int_0^L dz e^{-im\varphi} \sin\frac{n\pi z}{L} = \frac{-iV}{mn\pi^2}(1-(-)^n)(1-(-)^m) \\ &= \frac{-4iV}{mn\pi^2}\delta_{m,odd}\delta_{n,odd}, \qquad m \neq 0 \\ A_{0n} I_0(n\pi a/L) &= \frac{V}{L}\int_0^L dz \sin\frac{n\pi z}{L} = \frac{V}{n\pi}(1-(-)^n) = \frac{2V}{n\pi}\delta_{n,odd} \end{aligned}$$

Then

$$\phi = \sum_{n=1,odd}^{\infty} \frac{2V}{n\pi}\sin\frac{n\pi z}{L}\frac{I_0(n\pi\rho/L)}{I_0(n\pi a/L)} - \sum_{m,odd}\sum_{n=1,odd}^{\infty}\frac{4iV}{mn\pi^2}e^{im\varphi}\sin\frac{n\pi z}{L}\frac{I_m(n\pi\rho/L)}{I_m(n\pi a/L)}$$

20. J, Problem 3.12, parts a),b),c). Note that the answer to part a) can be expressed as an integral in one variable only. You may consult tables of integrals involving Bessel functions, for example Gradshteyn and Ryzhik.

**Solution**:

a) Since $\rho, z$ are unbounded, we use $J_m(k\rho)e^{-kz}e^{im\varphi}$ with $0 < k < \infty$ continuous, when we write the general solution

$$\phi = \sum_m \int_0^{\infty} dk f_m(k) e^{im\varphi} J_m(k\rho) e^{-kz}$$

There is no $e^{+kz}$ dependence because we need good behavior as $z \to +\infty$. In general, orthogonality gives, from $\phi(\rho, \phi, z = 0) = V(\rho, \phi)$

$$f_m(k) = \frac{k}{2\pi}\int d\varphi \int \rho d\rho e^{-im\varphi} J_m(k\rho) V$$

But for this problem $V$ is independent of $\varphi$, so $f_m = 0$ unless $m = 0$, for which

$$f_0(k) = Vk\int_0^a \rho d\rho J_0(k\rho) = \frac{V}{k}\int_0^{ka} x dx J_0(x) = VaJ_1(ka)$$

where last equality uses $xJ_0 = (xJ_1)'$ which follows from Eq. (3.87,3.88) of Jackson. Then

$$\phi(z, \rho) = Va\int_0^{\infty} dk J_0(k\rho) J_1(ka) e^{-kz}$$

b) Above the center of the disk, $\rho = 0$ and

$$\phi(z,0) = Va\int_0^\infty dk J_1(ka)e^{-kz} = V\left[1 - \frac{z}{\sqrt{a^2+z^2}}\right]$$

where the integral can be found, e.g. in Gradshteyn and Ryzhik, p.707.

c)Above the edge of the disk, $\rho = a$ and we have

$$\begin{aligned}\phi(z,a) &= Va\int_0^\infty dk J_0(ka)J_1(ka)e^{-kz} = -V\int_0^\infty dk J_0(ka)\frac{dJ_0(ka)}{dk}e^{-kz} \\ &= -\frac{V}{2}\int_0^\infty dk \frac{dJ_0^2(ka)}{dk}e^{-kz} = \frac{V}{2}\left(1 - \frac{z}{a}\int_0^\infty dx J_0^2(x)e^{-xz/a}\right)\end{aligned}$$

Consulting G-R, p. 709, $\int_0^\infty dx J_0^2 e^{-\alpha x} = 2K(2/\sqrt{4+\alpha^2})/\pi\sqrt{4+\alpha^2}) \equiv kK(k)/\pi$, with $k = 2/\sqrt{4+\alpha^2}$. For our case $\alpha = z/a$ and

$$\phi = \frac{V}{2}\left(1 - \frac{kz}{\pi a}K(k)\right), \qquad k = \frac{2a}{\sqrt{z^2+4a^2}}$$