Electromagnetic Theory I

Solution Set 6

Due: 14 October 2020

21. a) Use the power series definition of the Bessel function

$$J_m(x) \equiv \sum_{k=0}^{\infty} \frac{(-)^k}{k!\Gamma(k+1+m)} \left(\frac{x}{2}\right)^{2k+m} \tag{1}$$

to derive the recursion formulas for Bessel functions:

$$J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x), \qquad J_{m-1}(x) - J_{m+1}(x) = 2\frac{dJ_m(x)}{dx} \tag{2}$$

Solution: The power series for J_{m-1} and J_{m+1} can be written

$$\begin{aligned}
J_{m-1}(x) &\equiv \sum_{k=0}^{\infty} \frac{(-)^k}{k!\Gamma(k+m)} \left(\frac{x}{2}\right)^{2k+m-1} \\
J_{m+1}(x) &\equiv \sum_{k=0}^{\infty} \frac{(-)^k}{k!\Gamma(k+m+2)} \left(\frac{x}{2}\right)^{2k+m+1} = \sum_{k=1}^{\infty} \frac{-(-)^k}{(k-1)!\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2k+m-1}
\end{aligned}$$

Then

$$\begin{aligned}
J_{m-1} \pm J_{m+1} &= \frac{1}{\Gamma(m)} \left(\frac{x}{2}\right)^{m-1} + \sum_{k=1}^{\infty} \frac{(-)^k(k+m\mp k)}{k!\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2k+m-1} \\
&= \sum_{k=0}^{\infty} \frac{(-)^k}{k!\Gamma(k+m+1)} \begin{bmatrix} m \\ 2k+m \end{bmatrix} \left(\frac{x}{2}\right)^{2k+m-1} \\
&= \begin{bmatrix} 2m/x \\ 2d/dx \end{bmatrix} \sum_{k=0}^{\infty} \frac{(-)^k}{k!\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2k+m} = \begin{bmatrix} 2m/x \\ 2d/dx \end{bmatrix} J_m(x)
\end{aligned}$$

b) Explain why these same recursion formulas are valid for $N_m, H_m^{(1)}, H_m^{(2)}$.

Solution: From the definition of N_m,

$$\begin{aligned}
N_{m\pm1} &= \frac{J_{m\pm1}\cos(m\pm1)\pi - J_{-m\mp1}}{\sin(m\pm1)\pi} = \frac{J_{m\pm1}\cos m\pi + J_{-m\mp1}}{\sin m\pi} \\
N_{m-1} \pm N_{m+1} &= (J_{m-1} \pm J_{m+1})\cot m\pi + (J_{-m+1} \pm J_{-m-1})\csc m\pi \\
&= \begin{bmatrix} 2m/x \\ 2d/dx \end{bmatrix} J_m \cot m\pi \pm \begin{bmatrix} -2m/x \\ 2d/dx \end{bmatrix} J_{-m} \csc m\pi \\
&= \begin{bmatrix} 2m/x \\ 2d/dx \end{bmatrix} (J_m \cot m\pi - J_{-m}\csc m\pi) = \begin{bmatrix} 2m/x \\ 2d/dx \end{bmatrix} N_m
\end{aligned} \tag{3}$$

Establishing the identities for N_m. But then since $H_m^{(1),(2)} = J_m \pm iN_m$ are linear combinations of J_m, N_m the identities immediately apply to them as well.

c) Using the definitions

$$I_m(x) = i^{-m}J_m(ix), \qquad K_m(x) = \frac{\pi i^{m+1}}{2}H_m^{(1)}(ix) \tag{4}$$

obtain the analogous recursion formulas for I_m, K_m.

Solution: We simply write out

$$\begin{aligned}
I_{m-1}(x) \pm I_{m+1}(x) &= i^{-m+1}J_{m-1}(ix) \pm i^{-m-1}J_{m+1}(ix) = i^{-m+1}(J_{m-1}(ix) \mp J_{m+1}(ix)) \\
&= i^{-m+1}\begin{bmatrix} 2d/d(ix) \\ 2m/ix \end{bmatrix} J_m(ix) = \begin{bmatrix} 2d/dx \\ 2m/x \end{bmatrix} I_m(x) \\
K_{m-1}(x) \pm K_{m+1}(x) &= \frac{\pi}{2}(i^m H_{m-1}^{(1)}(ix) \pm i^{m+2}H_{m+1}^{(1)}(ix)) = \frac{\pi}{2}i^m(H_{m-1}^{(1)}(ix) \mp H_{m+1}^{(1)}(ix)) \\
&= \frac{\pi}{2}i^m \begin{bmatrix} 2d/d(ix) \\ 2m/ix \end{bmatrix} H_m^{(1)}(ix) = -\begin{bmatrix} 2d/dx \\ 2m/x \end{bmatrix} K_m(x)
\end{aligned}$$

Notice the sign differences for I, K compared to the others.

22. In class, we obtained the empty space Green function in cylindrical coordinates in the form

$$\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{4\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk J_{|m|}(k\rho)J_{|m|}(k\rho')e^{im(\varphi-\varphi')}e^{-k|z-z'|} \tag{5}$$

a) By adjusting the z dependent factors in this formula obtain the Dirichlet Green function for the region between the two infinite parallel planes $z = 0$ and $z = L$.

Solution: We introduce the notation $z_<, z_>$. Then we consider the product

$$\begin{aligned}
4\sinh kz_< \sinh k(L-z_>) &= e^{k(L+z_<-z_>)} + e^{-k(L+z_<-z_>)} - e^{k(L-z_<-z_>)} - e^{-k(L-z_<-z_>)} \\
&= e^{k(L+z_<-z_>)} + e^{-k(L+z_<-z_>)} - e^{k(L-z-z')} - e^{-k(L-z-z')} \\
&= (e^{kL} - e^{-kL})e^{-k(z_>-z_<)} + e^{-kL}2\cosh k(z-z') - 2\cosh k(L-z-z') \\
\frac{2\sinh kz_< \sinh k(L-z_>)}{\sinh kL} &= e^{-k|z-z'|} + \frac{e^{-kL}\cosh k(z-z') - \cosh k(L-z-z')}{\sinh kL}
\end{aligned}$$

The left side of the last equation is zero on the planes $z = 0, L$, satisfies the Laplace equation for $Z(z)$ when $z \neq z'$, and the equality shows it has the same discontinuity at $z = z'$ as $e^{-k|z-z'|}$. We conclude that

$$G_D = \frac{1}{4\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk J_{|m|}(k\rho)J_{|m|}(k\rho')e^{im(\varphi-\varphi')}\frac{2\sinh kz_< \sinh k(L-z_>)}{\sinh kL} \tag{6}$$

b) Use this Green function to calculate the potential between the planes when the plane $z = 0$ is held at zero potential and on the plane $z = L$ the potential is $\phi = V$ for $\rho \le a$ and $\phi = 0$ for $\rho \ge a$. Your answer will be an integral over k.

Solution: To apply Green's theorem, we need

$$\begin{aligned}
\hat{n} \cdot \nabla' G_D \Big|_{z'=L} &= \frac{\partial G_D}{\partial z'} = \frac{1}{4\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dk J_{|m|}(k\rho) J_{|m|}(k\rho') e^{im(\varphi-\varphi')} \frac{-2k \sinh kz}{\sinh kL} \\
\phi &= -V \int_0^a \rho' d\rho' \int_0^{2\pi} d\varphi \frac{1}{4\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dk J_{|m|}(k\rho) J_{|m|}(k\rho') e^{im(\varphi-\varphi')} \frac{-2k \sinh kz}{\sinh kL} \\
&= V \int_0^\infty dk J_0(k\rho) \frac{k \sinh kz}{\sinh kL} \int_0^a \rho' d\rho' J_0(k\rho')
\end{aligned}$$

Next we use $xJ_0(x) = d(xJ_1)/dx$ to do the ρ' integral

$$\begin{aligned}
\int_0^a \rho' d\rho' J_0(k\rho') &= \frac{1}{k^2} \int_0^{ka} dx \frac{d}{dx}(xJ_1(x)) = \frac{a}{k} J_1(ka) \\
\phi &= V \int_0^\infty a dk J_0(k\rho) J_1(ka) \frac{\sinh kz}{\sinh kL} = V \int_0^\infty dx J_0(x\rho/a) J_1(x) \frac{\sinh xz/a}{\sinh xL/a}
\end{aligned}$$

c) Check that your answer reduces to the expected results when $a \to \infty$ at fixed a, ρ, z and also when $L \to \infty$ at fixed $a, \rho, L - z$.

Solution: When $a \to \infty$, $J_0(x\rho/a) \to J_0(0) = 1$; and $\sinh xz/a / \sinh xL/a \to z/L$, so that

$$\phi \to \frac{zV}{L} \int_0^\infty dx j_1(x) = -\frac{zV}{L} \int_0^\infty dx J_0'(x) = \frac{zV}{L} \tag{7}$$

which is the potential in the region between a grounded plane at $z = 0$ and one at $z = L$ held at potential V.

When $L, z \to \infty$ with $L - z < a, \rho$ fixed, the ratio of sinh's can be rearranged:

$$\frac{\sinh(xz/a)}{\sinh(xL/a)} = e^{-(L-z)x/a} \left(1 + \frac{e^{-2xL/a} - e^{-2xz/a}}{1 - e^{-2xL/a}} \right) \tag{8}$$

The 1 term in parentheses $e^{-x(L-z)/a}$ just reproduces the results of Problem 3.12, with $L - z$ playing the role of z:

$$\phi \to V \int_0^\infty dx J_0(x\rho/a) J_1(x) e^{-x(L-z)/a} = Va \int_0^\infty dk J_0(k\rho) J_1(ka) e^{-k(L-z)} \tag{9}$$

The corrections all exponentially improve the convergence of the integral at large x.

23. Setting $z' = h < L$ and $\rho' = 0$, we know that q/ϵ_0 times the Green function obtained in part a) of the previous problem is just the potential for the system of two infinite grounded conducting planes with a charge q placed on the z axis at $z = h$.

a) Calculate the surface charge density induced on the upper plate as a function of ρ.

Solution: With $\rho' = 0$, $J_m(k\rho') \to \delta_{m0}$ so the potential reduces to

$$\phi = \frac{q}{4\pi\epsilon_0}\int_0^\infty dk J_0(k\rho)\frac{2\sinh kz_< \sinh k(L-z_>)}{\sinh kL} \tag{10}$$

where $z_<$ ($z_>$) is the smaller (larger) of h, z. The electric field at $z = L$ is

$$\begin{aligned}\boldsymbol{E} &= -\frac{\partial\phi}{\partial z}\hat{z} = \frac{q}{2\pi\epsilon_0}\int_0^\infty kdk J_0(k\rho)\frac{\sinh kh}{\sinh kL} \\ \sigma &= -\epsilon_0 E_z = -\frac{q}{2\pi}\int_0^\infty kdk J_0(k\rho)\frac{\sinh kh}{\sinh kL}\end{aligned}$$

b) Using Green's reciprocation theorem with the situation of the previous problem 22 as a comparison problem, show that the induced charge within a radius $\rho \leq a$ on the upper plate is $Q(a) = -q\phi'(h,0)/V$ where ϕ' is the potential you found in part b) of problem 22. From this result determine the total charge induced on the upper plate.

Solution: Let us call the charge and surface density and potential of the situation of problem 22 σ', ϕ' and the corresponding quantities for the present problem q, σ, ϕ. Then the reciprocation theorem states that

$$q\phi'(h,0) + V\int_{\text{Disk}} dA\sigma = 0 \tag{11}$$

The right side is zero because all the charge in problem 22 is located on the grounded planes of the present problem. The integral in the second term on the left is just the charge within a radius $\rho \leq a$ on the upper plate of the present problem. The equation tells us that this charge is $-q\phi'(h,0)/V$. As we saw in problem 22 c), in the limit $a \to \infty$, $\phi'(z,\rho) \to zV/L$. Thus the total charge on the upper plate of the current problem is $-qh/L$.

c) We also obtained an alternative expression for the empty space Green function obtained in cylindrical coordinates:

$$\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{(2\pi)^2}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} dk I_{|m|}(|k|\rho_<)K_{|m|}(|k|\rho_>)e^{im(\varphi-\varphi')}e^{ik(z-z')}. \tag{12}$$

By adjusting the z-dependent factor obtain the corresponding alternate Dirichlet Green function for the region between two parallel planes. Note that the adjustment includes replacing the integral over k by a discrete sum!

Solution: To arrange Dirichlet conditions at $z = 0, L$ we must replace $e^{ik(z-z')} \to C\sin(n\pi z/L)\sin(n\pi z'/L)$. And the integral over k must be restricted to a sum over the discrete values $k = n\pi/L$, $n = 1, 2, \cdots$. C must be chosen so that

$$\sum_{n=1}^{\infty} C\sin(n\pi z/L)\sin(n\pi z'/L) = 2\pi\delta(z - z') \qquad (13)$$

for $z, z' \in (0, L)$. Thus $C = 4\pi/L$ so

$$G_D = \frac{1}{\pi L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} I_{|m|}(n\pi\rho_</L)K_{|m|}(n\pi\rho_>/L)e^{im(\varphi-\varphi')}\sin(n\pi z/L)\sin(n\pi z'/L)$$

d) Using part c) write an alternate expression for the potential of this problem and calculate the surface charge density induced on both plates. Integrate to find the total induced charge and compare to that obtained in part b).

Solution: To get the potential we multiply G_D by q/ϵ_0 and set $\rho_< = 0, \rho_> = \rho, z' = h$:

$$\phi = \frac{q}{\pi\epsilon_0 L}\sum_{n=1}^{\infty}\sin\frac{n\pi h}{L}\sin\frac{n\pi z}{L}K_0(n\pi\rho/L)$$

where we used $I_0(0) = 1, I_m(0) = 0$ for $m > 0$. The surface charge densities on the two plates are

$$\begin{aligned}
\sigma_0(\rho) &= \epsilon_0 E_z = -\epsilon_0\frac{\partial\phi}{\partial z}\bigg|_{z=0} = -\frac{q}{L^2}\sum_{n=1}^{\infty} n\sin\frac{n\pi h}{L}K_0(n\pi\rho/L) \\
\sigma_L(\rho) &= -\epsilon_0 E_z = \epsilon_0\frac{\partial\phi}{\partial z}\bigg|_{z=L} = \frac{q}{L^2}\sum_{n=1}^{\infty} n(-)^n\sin\frac{n\pi h}{L}K_0(n\pi\rho/L) \qquad (14)
\end{aligned}$$

The charges on the two plates are

$$\begin{aligned}
Q_L &= 2\pi\int_0^{\infty}\rho d\rho\sigma_L(\rho) = \frac{2\pi q}{L^2}\sum_{n=1}^{\infty} n(-)^n\sin\frac{n\pi h}{L}\int_0^{\infty}\rho d\rho K_0(n\pi\rho/L) \\
&= \frac{2q}{\pi}\sum_{n=1}^{\infty}\frac{(-)^n}{n}\sin\frac{n\pi h}{L}\int_0^{\infty} x dx K_0(x) = -\frac{2q}{\pi}\sum_{n=1}^{\infty}\frac{(-)^n}{n}\sin\frac{n\pi h}{L}\int_0^{\infty} dx\frac{d}{dx}(xK_1(x)) \\
&= \frac{2q}{\pi}\sum_{n=1}^{\infty}\frac{(-)^n}{n}\sin\frac{n\pi h}{L} = \frac{q}{\pi i}\sum_{n=1}^{\infty}\frac{(-)^n}{n}(e^{in\pi h/L} - e^{-in\pi h/L}) \\
&= -\frac{q}{\pi i}\ln\frac{1 + e^{i\pi h/L}}{1 + e^{-i\pi h/L}} = -\frac{q}{\pi i}\ln e^{i\pi h/L} = -\frac{qh}{L} \\
Q_0 &= -\frac{2q}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi h}{L} = -\frac{q}{\pi i}\sum_{n=1}^{\infty}\frac{1}{n}(e^{in\pi h/L} - e^{-in\pi h/L}) \\
&= \frac{q}{\pi i}\ln\frac{1 - e^{i\pi h/L}}{1 - e^{-i\pi h/L}} = \frac{q}{\pi i}\ln\frac{1 + e^{i\pi(h-L)/L}}{1 + e^{-i\pi(h-L)/L}} = \frac{q}{\pi i}\ln e^{i\pi(h-L/L} = -\frac{q(L-h)}{L}
\end{aligned}$$

The result for Q_L agrees with part b), and the total charge on both plates is $Q_0+Q_L = -q$, in accord with Gauss Law.

24. Solve J, Problem 3.22.

Solution: The Laplacian in polar coordinates is just $\nabla^2 = \partial^2/\partial\rho^2+(1/\rho)\partial/\partial\rho+(1/\rho^2)\partial^2/\partial\varphi^2$. For this problem the azimuthal separation factor should be $\sin(m\pi\varphi/\beta)$, so that it vanishes at $\varphi = 0, \beta$. Then the radial factor must satisfy

$$-\frac{\partial^2}{\partial\rho^2}R - \frac{1}{\rho}\frac{\partial}{\partial\rho}R + \frac{m^2\pi^2}{\beta^2\rho^2}R = 0 \tag{15}$$

which is solved by the power ρ^p with $p^2 = m^2\pi^2/\beta^2$ or $p = \pm m\pi/\beta$. When we construct the Green function using the usual notation $\rho_<, \rho_>$, we should choose the positive powers for $\rho_<$. For this problem $\rho \leq a$ so both positive and negative powers are allowed for $\rho_>$. Indeed we must take a linear combination of the two to make the radial function vanish at $\rho_> = a$. Combining all the information so far we have learned that

$$G_D(\boldsymbol{\rho}, \boldsymbol{\rho}') = \sum_{m=1}^{\infty} A_m \rho_<^{m\pi/\beta}\left(\frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}}\right)\sin\frac{m\pi\varphi}{\beta}\sin\frac{m\pi\varphi'}{\beta} \tag{16}$$

A_m is fixed by the normalizing the discontinuity so that Laplacian produces the radial delta function:

$$\begin{aligned}
-\frac{d^2}{d\rho^2}\frac{\rho_<^{m\pi/\beta}}{\rho_>^{m\pi/\beta}} &= -\frac{d^2}{d\rho^2}\left(\theta(\rho-\rho')\frac{\rho'^{m\pi/\beta}}{\rho^{m\pi/\beta}} + (\theta(\rho'-\rho)\frac{\rho^{m\pi/\beta}}{\rho'^{m\pi/\beta}}\right) \\
&= -\frac{d}{d\rho}\left(\theta(\rho-\rho')(-m\pi/\beta)\frac{\rho'^{m\pi/\beta}}{\rho^{m\pi/\beta+1}} + (\theta(\rho'-\rho)(m\pi/\beta)\frac{\rho^{m\pi/\beta-1}}{\rho'^{m\pi/\beta}}\right) \\
&= \frac{\delta(\rho-\rho')}{\rho}\frac{2m\pi}{\beta} + \cdots
\end{aligned} \tag{17}$$

Thus

$$-\nabla^2 G = \frac{\delta(\rho-\rho')}{\rho}\sum_{m=1}^{\infty}\frac{2mA_m\pi}{\beta}\sin\frac{m\pi\varphi}{\beta}\sin\frac{m\pi\varphi'}{\beta} = \frac{\delta(\rho-\rho')}{\rho}\delta(\varphi-\varphi') \tag{18}$$

The last equality requires $mA_m\pi = 1$, so

$$G_D(\boldsymbol{\rho}, \boldsymbol{\rho}') = \sum_{m=1}^{\infty}\frac{1}{m\pi}\rho_<^{m\pi/\beta}\left(\frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}}\right)\sin\frac{m\pi\varphi}{\beta}\sin\frac{m\pi\varphi'}{\beta} \tag{19}$$

Here, as always, we differ from Jackson by a factor of $1/4\pi$ due to our different Green function equations.