

# Electromagnetic Theory I

## Solution Set 7

Due: 28 October 2020

25. J, Problem 4.1

**Solution:** this problem is an exercise in calculating the multipole moments of simple point charge distributions in spherical basis, where

$$q_{lm} = \int d^3x \rho(\mathbf{r}) r^l Y_{lm}^*(\theta, \varphi). \quad (1)$$

Since the charges are separated by finite distances, neither of these arrangements is a pure multipole.

a) Here  $\rho = q\delta(\rho - a)\delta(z)[\delta(\varphi) + \delta(\varphi - \pi/2) - \delta(\varphi - \pi) - \delta(\varphi - 3\pi/2)]\delta(z)/\rho$

$$\begin{aligned} q_{lm} &= qa^l [Y_{lm}^*(\pi/2, 0) + Y_{lm}^*(\pi/2, \pi/2) - Y_{lm}^*(\pi/2, \pi) - Y_{lm}^*(\pi/2, 3\pi/2)] \\ &= qa^l \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(0) (1 - (-)^m) (1 + (-i)^m) \\ &= qa^l \frac{(-)^{(l+m)/2}}{2^l l!} (1 - i^m) \sqrt{\frac{(2l+1)(l-m)!(l+m)!}{4\pi}} \binom{l}{(l+m)/2} 2\delta_{m,\text{odd}} \delta_{l,\text{odd}} \end{aligned}$$

Here we used the  $x \rightarrow 0$  limit of  $P_l^m$ :

$$P_l^m(x) \rightarrow \frac{(-)^m}{2^l l!} \frac{d^{l+m}}{dx^{l+m}} \sum_{n=0}^l x^{2n} (-)^{l-n} \binom{l}{n} \rightarrow \frac{(-)^{(l+m)/2}}{2^l l!} \delta_{l+m,\text{even}} (l+m)! \binom{l}{(l+m)/2}$$

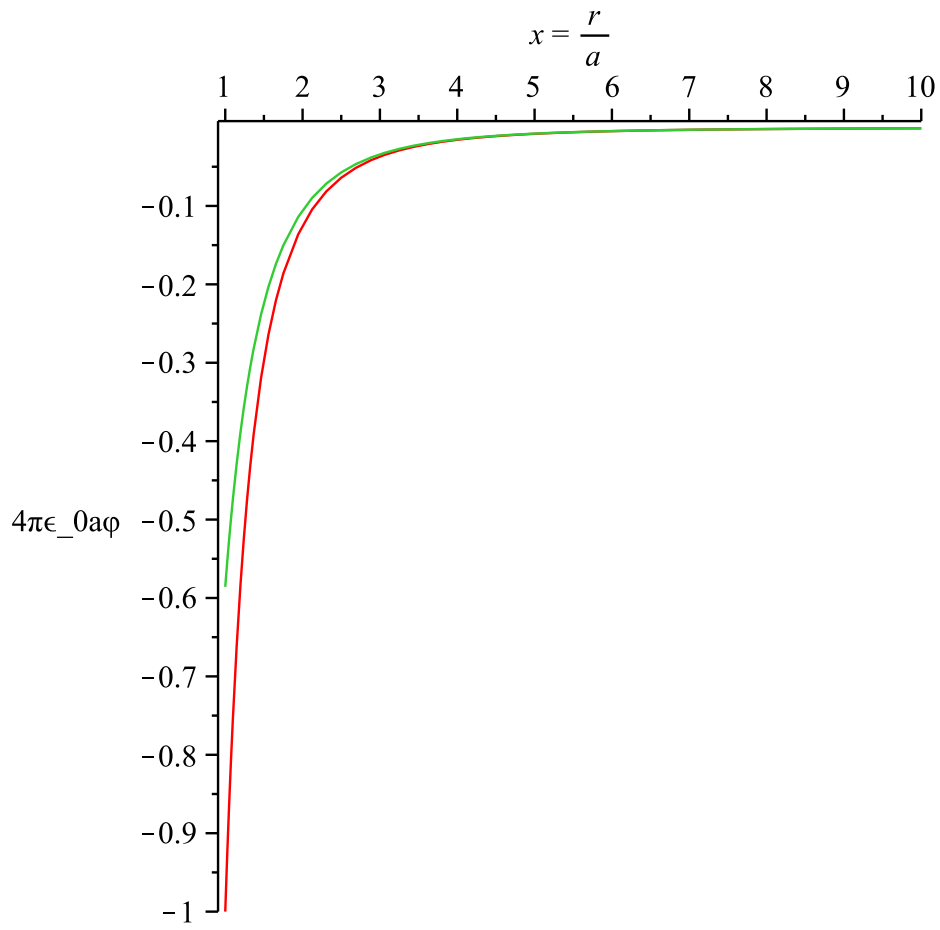
b)

$$\begin{aligned} q_{lm} &= qa^l (Y_{lm}^*(0, \varphi) + Y_{lm}^*(\pi, \varphi) - 2qY_{00}^* \delta_{l0} \delta_{m0}) \\ &= 2qa^l \delta_{m0} (\delta_{l,\text{even}} - \delta_{l0}) \sqrt{\frac{2l+1}{4\pi}} \end{aligned}$$

c)

$$\begin{aligned} \phi &= \frac{1}{\epsilon_0} \sum_{k=1}^{\infty} \frac{2qa^{2k}}{4k+1} \sqrt{\frac{4k+1}{4\pi}} \frac{Y_{2k,0}}{r^{2k+1}} = \frac{q}{2\pi\epsilon_0} \sum_{k=1}^{\infty} \frac{a^{2k} P_{2k}}{r^{2k+1}} \\ &= \frac{q}{4\pi\epsilon_0} \frac{a^2}{r^3} (3 \cos^2 \theta - 1) + \dots \end{aligned}$$

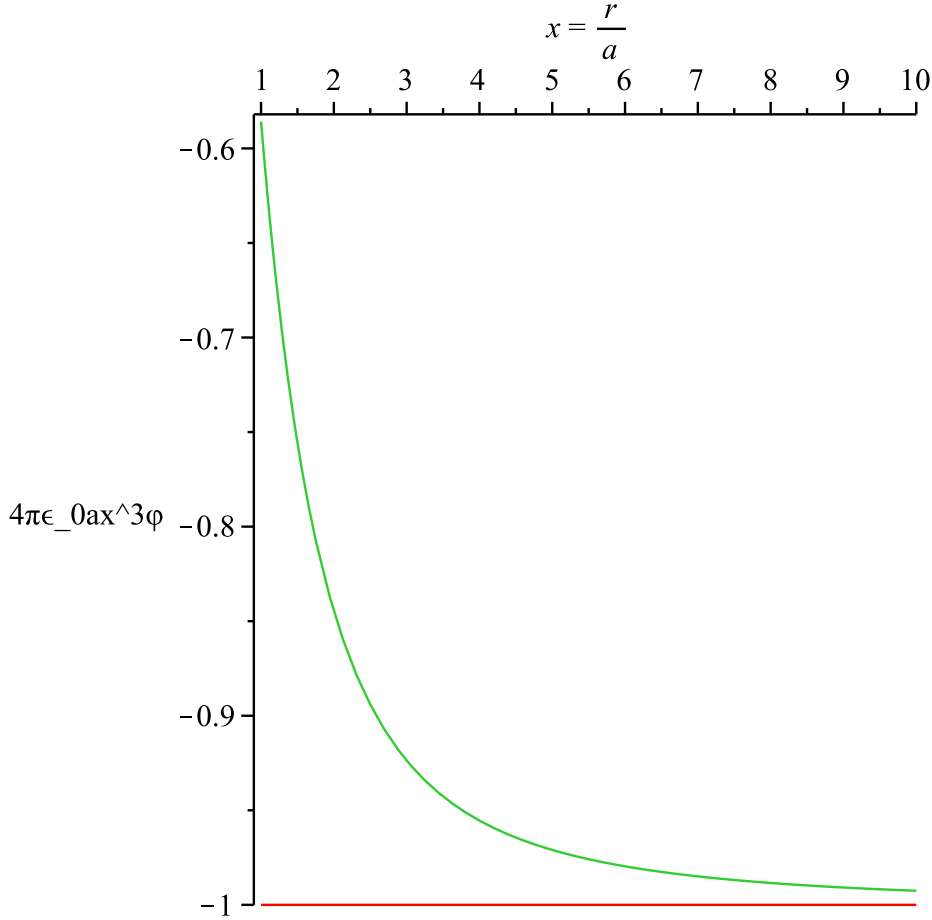
At  $\theta = \pi/2$ ,  $\phi \sim -qa^2/(4\pi\epsilon_0 r^3)$ . The lower curve in the following plot is this leading term in the  $xy$ -plane ( $\theta = \pi/2$ ). The upper curve is the exact potential.



d)

$$\phi(r, \theta = \pi/2) = \frac{q}{2\pi\epsilon_0} \left( \frac{1}{\sqrt{r^2 + a^2}} - \frac{1}{r} \right) \sim -\frac{qa^2}{4\pi\epsilon_0 r^3} + \dots$$

We plot the exact potential times  $r^3$  and leading multipole times  $r^3$  in the  $xy$ -plane in the following figure:



26. We have seen that in Cartesian basis the  $2^l$  pole moment is a rank  $l$  tensor  $Q^{i_1 i_2 \dots i_l}$ , which is completely symmetric in its indices and traceless in every pair of indices. In this problem we repeat the counting we did in class in terms of the generating function defined in part a) below and then find the generalization to  $d$  dimensions.

- a) Show that any rank  $l$  tensor with complete symmetry has  $N_l = \frac{(l+1)(l+2)}{2}$  independent components by finding the generating function  $p_3(x) = \sum_{l=0}^{\infty} N_l x^l$ . Since  $Q$  is symmetric in all its indices, all components in which  $n_1$  indices have the value 1,  $n_2$  indices have the value 2 and  $n_3$  indices have the value 3 are equal. Thus the number of independent components is the number of ways you can partition  $l$  into the sum of three integers  $l = n_1 + n_2 + n_3$  where each  $n_k$  is a nonnegative integer  $\leq l$ . Using these

facts, prove that  $p_3(x) = (1 - x)^{-3}$  and obtain the desired result by developing  $p(x)$  in a Taylor series.

**Solution:** Since  $Q^{i_1 i_2 \dots i_l}$  is symmetric in all its indices we may label the independent components with the indices ordered:  $i_1 \leq i_2 \leq i_3 \leq \dots \leq i_l$ . Each index can have three values  $i_k = 1, 2, 3$ . Let the first  $n_1$  indices have value 1, the next  $n_2$  have value 2 and the last  $n_3$  have the value 3. Then  $n_1 + n_2 + n_3 = l$ , and each  $n_i$  ranges from 0 to  $l$ . Clearly the coefficient of  $x^l$  in the function  $p_3(x) = \sum_{n_1, n_2, n_3=0}^{\infty} x^{n_1+n_2+n_3}$  counts the number of ways that  $n_1 + n_2 + n_3 = l$ , and hence the number of independent components of  $Q$  disregarding the traceless conditions.

$$\begin{aligned} p_3(x) &= \sum_{n_1, n_2, n_3=0}^{\infty} x^{n_1+n_2+n_3} = \left( \sum_{n=0}^{\infty} x^n \right)^3 = (1-x)^{-3} = \sum_{l=0}^{\infty} \binom{-3}{l} (-x)^l \\ N_l &= (-)^l \binom{-3}{l} = (-)^l \frac{-3(-4) \dots (-2-l)}{l!} = \frac{(l+2)(l+1)}{2} \end{aligned} \quad (2)$$

- b) The tracelessness condition gives  $N_{l-2}$  relations among these components. Use this counting to prove that  $Q^{i_1 i_2 \dots i_l}$  has  $2l + 1$  independent components.

**Solution:** Imposing  $N_{l-2}$  traceless conditions gives

$$N_l - N_{l-2} = \frac{(l+2)(l+1) - l(l-1)}{2} = 2l + 1 \quad (3)$$

as desired.

- c) Now generalize to an arbitrary spatial dimension  $d$  in which you can show that  $p_d(x) = (1 - x)^{-d}$ . Find the number of components of a symmetric tensor first, and then by subtracting find the number of components of a traceless one.

**Solution:**

$$\begin{aligned} p_d(x) &= \sum_{n_1, \dots, n_d=0}^{\infty} x^{n_1+\dots+n_d} = \left( \sum_{n=0}^{\infty} x^n \right)^d = (1-x)^{-d} = \sum_{l=0}^{\infty} \binom{-d}{l} (-x)^l \\ N_l^d &= (-)^l \binom{-d}{l} = (-)^l \frac{-d(-d-1) \dots (-d+1-l)}{l!} \\ &= \frac{(d+l-1)(d+l-2) \dots (l+1)}{(d-1)!} \\ N_l^d - N_{l-2}^d &= \frac{(d+l-3) \dots (l+1)}{(d-1)!} [(d+l-1)(d+l-2) - l(l-1)] \\ &= \frac{(d+l-3) \dots (l+1)}{(d-2)!} [2l+d-2] = \frac{(d+l-3)!}{(d-2)!} [2l+d-2] \end{aligned} \quad (4)$$

Notice that for  $d = 2$ , this reduces to precisely 2 independent components for every  $l$ !

27. J, Problem 4.4

**Solution:**

a) The  $2^l$  moment about a point  $\mathbf{r}_0$  has the form

$$\begin{aligned} Q_{\mathbf{r}_0}^{i_1 \dots i_l} &= A_l \int d^3 r [(r - r_0)^{i_1} \dots (r - r_0)^{i_l} \text{Traces}] \rho(\mathbf{r}') \\ &= Q_{\mathbf{0}}^{i_1 \dots i_l} - r_0^{i_1} Q_{\mathbf{0}}^{i_2 \dots i_l} - r_0^{i_2} Q_{\mathbf{0}}^{i_1 i_3 \dots i_l} + \dots \end{aligned} \quad (5)$$

where the dots signify terms of higher order in  $r_0$  as well as terms with different indices on  $r_0$ . But if the coefficients of the positive powers of  $r_0$  are proportional to a lower multipole moment than the  $2^l$  moment and  $l$  is the lowest nonvanishing moment all these terms are zero and we would have  $Q_{\mathbf{r}_0}^{i_1 \dots i_l}$  is independent of  $r_0$ . It is clear that those coefficients involve an integral of powers of  $r^i$  times the charge density, but it is not obvious that the coefficients are traceless in all pairs of indices, as the multipole moments must be. To see that they are actual multipole moments it is better to consider alternative multipole expansions of the potential due to the charge distribution:

$$\begin{aligned} \phi &= \frac{1}{4\pi\epsilon_0} \int d^3 x' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi\epsilon_0} \int d^3 x' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}_0 - (\mathbf{r}' - \mathbf{r}_0)|} \\ \sum_{n=0}^{\infty} \frac{r^{i_1} \dots r^{i_n} Q_{\mathbf{r}_0}^{i_1 \dots i_n}}{n! r^{2n+1}} &= \sum_{n=0}^{\infty} \frac{(r - r_0)^{i_1} \dots (r - r_0)^{i_n} Q_{\mathbf{r}_0}^{i_1 \dots i_n}}{n! |\mathbf{r} - \mathbf{r}_0|^{2n+1}} \end{aligned} \quad (6)$$

Now take  $\mathbf{r} \rightarrow \infty$  on both sides. Then if  $n = l$  is the lowest nonvanishing moment, the equality tends to

$$\frac{r^{i_1} \dots r^{i_l} Q_{\mathbf{r}_0}^{i_1 \dots i_l}}{l! r^{2l+1}} = \frac{r^{i_1} \dots r^{i_l} Q_{\mathbf{r}_0}^{i_1 \dots i_l}}{l! r^{2l+1}} \quad (7)$$

which implies  $Q_{\mathbf{r}_0}^{i_1 \dots i_l} = Q_{\mathbf{r}_0}^{i_1 \dots i_l}$ . By expanding both sides of the equality in powers of  $\mathbf{r} - \mathbf{r}_0$ , one sees that the higher nonvanishing multipole  $Q_{\mathbf{r}_0}^{i_1 \dots i_n}$  is equal to  $Q_{\mathbf{r}_0}^{i_1 \dots i_n}$  + powers of  $\mathbf{r}_0$  times lower multipole moments.

b) We simply write out definitions:

$$\begin{aligned} q' &= \int d^3 r \rho = q \\ \mathbf{p}' &= \int d^3 r (\mathbf{r} - \mathbf{R}) \rho = \mathbf{p} - \mathbf{R}q \\ Q^{ij} &= \int d^3 r [3(r^i - R^i)(r^j - R^j) - \delta_{ij}(\mathbf{r} - \mathbf{R})^2] \rho \\ &= Q^{ij} - 3(p^i R^j + p^j R^i) + 2\delta_{ij} \mathbf{R} \cdot \mathbf{p} + (3R^i R^j - \delta_{ij} \mathbf{R}^2) q \end{aligned} \quad (8)$$

- c) Clearly choosing  $\mathbf{R} = \mathbf{p}/q$  make  $\mathbf{p}' = 0$  whatever  $\mathbf{p}$  is. However  $Q'^{ij} = 0$  would imply that the original  $Q^{ij} = 3(p^i R^j + p^j R^i) - 2\delta_{ij} \mathbf{R} \cdot \mathbf{p} - (3R^i R^j - \delta_{ij} \mathbf{R}^2)q$ , which is not the most general form  $Q^{ij}$  could take. (There are only 3 independent components of  $\mathbf{R}$  but 5 independent components of  $Q^{ij}$ .)

28. We have found that the method of images with a single image charge can give the potential for a point charge outside a grounded conducting sphere, and also the fields for a charge above the planar interface between two different homogeneous dielectrics. But as we shall see, the method falls short for a point charge outside a uniform dielectric sphere. Assume the dielectric constant is  $\epsilon$  inside a sphere of radius  $R$  and  $\epsilon_0$  outside the sphere.

- a) Set up and solve for the potential of a point charge  $q$  a distance  $D > R$  from the center of the dielectric sphere as an expansion in Legendre polynomials (or spherical harmonics). You will have different expansion coefficients inside and outside the sphere, which you are to determine by the matching conditions at the boundary.

**Solution:** The potential outside the sphere is a superposition of the Coulomb potential of the external charge and an expansion in powers of  $1/r$ ; that inside is an expansion in powers of  $r$ :

$$\begin{aligned}\phi_{\text{out}} &= \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \left[ \frac{r_{<}^l}{r_{>}^{l+1}} + \frac{B_l}{r^{l+1}} \right] P_l(\cos\theta) \\ \phi_{\text{in}} &= \frac{q}{4\pi\epsilon} \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)\end{aligned}\quad (9)$$

where  $r_{<} = \min(r, D)$ . Continuity of  $D_n = \epsilon\partial\phi/\partial r$  and  $E_t \propto \partial\phi/\partial\theta$  at the boundary  $r = a$  lead to:

$$\frac{lR^{2l+1}}{D^{l+1}} = A_l R^{2l+1} + B_l(l+1), \quad \frac{R^{2l+1}}{D^{l+1}} = \frac{\epsilon_0}{\epsilon} A_l R^{2l+1} - B_l(l+1) \quad (10)$$

with solution:

$$B_l = \frac{l(\epsilon_0 - \epsilon)R^{2l+1}}{D^{l+1}[(l+1)\epsilon_0 + \epsilon l]}, \quad A_l = \frac{\epsilon(2l+1)}{D^{l+1}[(l+1)\epsilon_0 + \epsilon l]} \quad (11)$$

- b) Show that in the limit  $\epsilon/\epsilon_0 \rightarrow \infty$ , the result reduces to the potential for a point charge outside a conducting sphere, by expanding the latter solution in Legendre polynomials and comparing coefficients.

**Solution:** For  $\epsilon \rightarrow \infty$  and  $l \neq 0$ ,  $A_l/\epsilon \rightarrow 0$  and  $B_l \rightarrow -R^{2l+1}/D^{l+1}$ . But  $A_0/\epsilon = 1/\epsilon_0 D$  and  $B_0 = 0$  Thus  $\phi_{\text{in}} \rightarrow q/4\pi\epsilon_0 D$  and

$$\phi_{\text{out}} \rightarrow \frac{q}{4\pi\epsilon_0} \left[ \frac{1}{r_{>}} + \sum_{l=1}^{\infty} \left[ \frac{r_{<}^l}{r_{>}^{l+1}} - \frac{R^{2l+1}}{D^{l+1}r^{l+1}} \right] P_l(\cos\theta) \right] \quad (12)$$

On the other hand the potential for a charge outside a neutral conducting sphere is

$$\begin{aligned}
\phi &= \frac{q}{4\pi\epsilon_0} \left( \frac{1}{|\mathbf{r} - D\hat{z}|} - \frac{R/D}{|\mathbf{r} - R^2\hat{z}/D|} + \frac{R/D}{r} \right) \\
&= \frac{q}{4\pi\epsilon_0} \left( \sum_{l=0}^{\infty} \frac{r_{<}^l}{r_{>}^{l+1}} P_l - \frac{R}{D} \sum_{l=0}^{\infty} \frac{(R^2/D)^l}{r^{l+1}} P_l + \frac{R}{Dr} \right) \\
&= \frac{q}{4\pi\epsilon_0} \left( \sum_{l=0}^{\infty} \frac{r_{<}^l}{r_{>}^{l+1}} P_l - \sum_{l=1}^{\infty} \frac{R^{2l+1}}{D^{l+1}r^{l+1}} P_l \right)
\end{aligned} \tag{13}$$

which agrees with the  $\epsilon \rightarrow \infty$  limit of the solution for a dielectric sphere.

- c) For finite  $\epsilon/\epsilon_0$  discuss why the solution cannot be expressed in terms of a single image charge, even though when  $\epsilon/\epsilon_0 \rightarrow \infty$  it can.

**Solution:** Concentrating on the field outside the dielectric sphere we see that the coefficient  $B_l$  has an  $l$  dependent factor,  $l(\epsilon_0 - \epsilon)/[(l+1)\epsilon_0 + l\epsilon]$ . This  $l$  dependence prevents the sum over  $l$  from producing a simple Coulomb potential for some image charge. But when  $\epsilon \rightarrow \infty$  this factor tends to  $-1$  independent of  $l$ .