Electromagnetic Theory I

Solution Set 8

Due: 4 November 2020

29. J, Problem 4.10

Solution

a) The geometry of this problem is such that even though the physical situation is not isotropic, the isotropic ansatz $\boldsymbol{E} = E(r)\hat{r}$ can solve all of the equations for the problem. This is because a radial $\boldsymbol{E}$ field is tangential to the interface: the condition $\epsilon E_n^1 - \epsilon_0 E_n^2 = 0$ is automatically satisfied because $E_n = 0$ at the interface. Then Gauss' Law on a spherical surface of radius r determines $E(r)$:

$$\begin{aligned} Q &= \oint dAD_n = 2\pi r^2 \epsilon_0 E(r) + 2\pi r^2 \epsilon E(r) \\ \boldsymbol{E}(\boldsymbol{r}) &= \frac{Q}{2\pi(\epsilon_0 + \epsilon)r^2}\hat{r} \end{aligned} \tag{1}$$

b) The (free) surface charge density on the inner sphere is given by

$$\begin{aligned} \sigma_0 &= \epsilon E(a) = \frac{Q\epsilon}{2\pi(\epsilon_0 + \epsilon)a^2}, \qquad \text{on hemisphere under dielectric} \\ \sigma_\epsilon &= \epsilon_0 E(a) = \frac{Q\epsilon_0}{2\pi(\epsilon_0 + \epsilon)a^2}, \qquad \text{on hemisphere under vacuum} \end{aligned}$$

c) The polarization charge density on the inner surface of the dielectric is just $P_n = -(\epsilon - \epsilon_0)E(a)$, the negative sign coming because the normal should be outward with respect to the dielectric, i.e. $\hat{n} = -\hat{r}$. Thus

$$\sigma_{\text{bound}} = -\frac{Q(\epsilon - \epsilon_0)}{2\pi(\epsilon_0 + \epsilon)a^2} \tag{2}$$

on the inner surface of the dielectric. Notice that on this surface $\sigma_\epsilon + \sigma_{\text{bound}} = \sigma_0$, so the actual microscopic charge distribution ends up being uniform, explaining why the electric field induced by it was isotropic.

30. J, Problem 4.13

Solution: The gravitational force on the raised liquid $-\rho g \pi h(b^2 - a^2)\hat{z}$ must be balanced by the electrical force on the dielectric fluid. We get the latter by evaluating the electrostatic energy difference $\Delta U = \int d^3x(\epsilon - \epsilon_0)\boldsymbol{E}^2/2$. By Gauss' law the radial dependence of $\boldsymbol{E}$ is proportional to $1/r$ and the potential difference $V = -\int drE(r)$. Thus $E = -V/(r\ln(b/a))$. then

$$\Delta U = \frac{1}{2}(2\pi h)\int_a^b rdr(\epsilon - \epsilon_0)\frac{V^2}{r^2\ln^2(b/a)} = \frac{\pi h V^2}{\ln(b/a)}(\epsilon - \epsilon_0) \tag{3}$$

Since the energy is expressed at fixed V, the electric force is $\boldsymbol{F} = +\hat{z}\partial\Delta U/\partial h$. The equilibrium condition thus implies:

$$\epsilon - \epsilon_0 = \epsilon_0\chi_e = \frac{\rho g(b^2 - a^2)h\ln(b/a)}{V^2} \tag{4}$$

31. Consider the magnetic field produced by a current I in an infinitely long wire lying on the z axis $-\infty < z < \infty$.

a) Use symmetry arguments and Ampère's law to obtain the $\boldsymbol{B}$ field everywhere outside the wire. Express the Cartesian components of $\boldsymbol{B}$ as explicit functions of x, y, z.

Solution: Ampere's law with a circular loop centered on the z axis yields, with $r = \sqrt{x^2 + y^2}$,

$$\begin{aligned} \boldsymbol{B} &= \frac{\mu_0 I}{2\pi r}\hat{\varphi} = \frac{\mu_0 I}{2\pi r}(-\hat{x}\sin\varphi + \hat{y}\cos\varphi) \\ B_x &= -\frac{\mu_0 I y}{2\pi(x^2 + y^2)}, \qquad B_y = \frac{\mu_0 I x}{2\pi(x^2 + y^2)}, \qquad B_z = 0 \end{aligned}$$

b) By direct integration of each component of $\nabla \times \boldsymbol{A} = \boldsymbol{B}$, find the vector potential $\boldsymbol{A}$ for this $\boldsymbol{B}$ in Coulomb gauge, $\nabla \cdot \boldsymbol{A} = 0$.

Solution: We can find a vector potential with $A_x = A_y = 0$

$$\begin{aligned} \partial_y A_z &= B_x, \qquad A_z = -\frac{\mu_0 I}{4\pi}\ln(x^2 + y^2) + f(x, z) \\ B_y &= -\partial_x A_z = \frac{\mu_0 I x}{2\pi(x^2 + y^2)} + \partial_x f(x, z), \qquad \partial_x f(x, z) = 0 \end{aligned}$$

We may set $f = \textit{constant}$ by fixing a gauge $\nabla \cdot \boldsymbol{A} = \partial_z A_z = 0$. We can choose this constant to involve a distance r_0 to make the argument of the ln dimensionless

$$A_z = -\frac{\mu_0 I}{4\pi}\ln\frac{x^2 + y^2}{r_0^2}$$

c) Since $\nabla \times \boldsymbol{B} = 0$ "almost everywhere" we should be able to find a scalar potential such that $\boldsymbol{B} = -\nabla\phi$ "almost everywhere". By explicitly integrating the components of this equation, find a candidate for ϕ as an explicit function of x, y, z.

Solution:

$$
\begin{aligned}
-\frac{\partial\phi}{\partial z} &= 0, \qquad -\frac{\partial\phi}{\partial x} = B_x, \qquad \phi = \frac{\mu_0 I}{2\pi}\tan^{-1}\frac{x}{y} + f(y) \\
-\frac{\partial\phi}{\partial y} &= -f'(y) + \frac{\mu_0 I x}{2\pi(x^2+y^2)} = B_y, \qquad f'(y) = 0 \\
\phi &= \frac{\mu_0 I}{2\pi}\tan^{-1}\frac{x}{y} \qquad (5)
\end{aligned}
$$

d) In view of the fundamental theorem of calculus

$$
\phi(y) - \phi(x) = \int_x^y \boldsymbol{dl}\cdot\nabla\phi = -\int_x^y \boldsymbol{dl}\cdot\boldsymbol{B}, \qquad (6)
$$

explain how your result for part c) does not run afoul of Ampère's law.

Solution: The resolution of the puzzle is that $\tan^{-1}\frac{x}{y}$ is not a single valued function. To make this transparent use polar coordinates $x/y = \cot\varphi$ then $\tan^{-1}\frac{x}{y} = \pi/2 - \varphi$. So $\phi(r, \varphi + 2\pi) - \phi(r, \varphi) = -2\pi(\mu_0 I/2\pi) = -\mu_0 I$, in accord with Ampere's law.

32. J, Problem 5.7.

a) The distance from a point on the loop to a point on the z-axis is $\sqrt{a^2+z^2}$. Also $\boldsymbol{dl} = a\hat{\varphi}d\varphi$ and $\hat{\varphi}\times\boldsymbol{x} = z\hat{\rho} + a\hat{z}$, so

$$
\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\varphi a \frac{z\hat{\rho} + a\hat{z}}{(z^2+a^2)^{3/2}} = \frac{\mu_0 I}{4\pi}\frac{2\pi a^2\hat{z}}{(z^2+a^2)^{3/2}} = \frac{\mu_0 I a^2}{2}\frac{\hat{z}}{(z^2+a^2)^{3/2}} \qquad (7)
$$

b) Relocating the origin and adding the fields from the two coils gives

$$
\begin{aligned}
B_z &= \frac{\mu_0 I a^2}{2}\left(\frac{1}{((z+b/2)^2+a^2)^{3/2}} + \frac{1}{((z-b/2)^2+a^2)^{3/2}}\right) \\
&= \frac{\mu_0 I a^2}{2d^3}\left((1+z^2/d^2+bz/d^2)^{-3/2} + (1+z^2/d^2-bz/d^2)^{-3/2}\right) \\
&= \frac{\mu_0 I a^2}{2d^3}\sum_{n=0}^{\infty}\binom{-3/2}{n}\frac{b^n z^n}{d^{2n}}\left((1+z/b)^n + (-)^n(1-z/b)^n\right) \qquad (8)
\end{aligned}
$$

We need to keep terms up to $n = 4$. The $n = 0$ term in the sum is just 2. The $n = 1$ term is $-3z^2/d^2$. The $n = 2$ term is $(15/4)(b^2z^2/d^4)(1+z^2/b^2)$. The $n = 3$ term is $(-3\cdot5\cdot7/(3\cdot8))(b^3z^3/d^6)(3z/b) = (-3\cdot5\cdot7/8)(b^2z^4/d^6)$ and the $n = 4$ term is $(5\cdot7\cdot9/64)(b^4z^4/d^8)$, where we dropped powers of z higher than 4. Collecting terms we have for the sum

$$
\begin{aligned}
\Sigma &= 2 + z^2\frac{15b^2 - 12d^2}{4d^4} + 15z^4\frac{d^4 - 7b^2d^2/2 + 21b^4/16}{4d^8} \\
&= 2\left(1 + 3z^2\frac{b^2-a^2}{2d^4} + 15z^4\frac{b^4 - 6b^2a^2 + 2a^4}{16d^8}\right)
\end{aligned}
$$

which gives the quoted result.

$$B_z = \frac{\mu_0 I a^2}{d^3}\left(1 + 3z^2\frac{b^2-a^2}{2d^4} + 15z^4\frac{b^4-6b^2a^2+2a^4}{16d^8}\right) \tag{9}$$

c) We start with (243) in the lecture notes with $R \to a$ and expanded in powers of $ar\sin\theta/(r^2+a^2) = a\rho/(r^2+a^2)$

$$\begin{aligned}
\boldsymbol{A} &= \frac{Ia\mu_0}{4\pi}\hat{\varphi}\int d\varphi' \sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{(-2a\rho)^n\cos^{n+1}\varphi'}{(r^2+a^2)^{n+1/2}} \\
&= \frac{Ia\mu_0}{4\pi}\hat{\varphi}\sum_{n=\mathrm{odd}}\binom{-1/2}{n}\binom{n+1}{(n+1)/2}\frac{(-a\rho)^n}{(r^2+a^2)^{n+1/2}} \\
&= \frac{Ia\mu_0}{4\pi}\hat{\varphi}\left[\frac{a\rho}{(r^2+a^2)^{3/2}} + \binom{-1/2}{3}\binom{4}{2}\frac{-a^3\rho^3}{(r^2+a^2)^{7/2}} + \cdots\right] \\
&= \frac{Ia^2\mu_0}{4\pi}(-\hat{x}y+\hat{y}x)\left[\frac{1}{(r^2+a^2)^{3/2}} + \frac{15}{8}\frac{a^2\rho^2}{(r^2+a^2)^{7/2}} + \cdots\right]
\end{aligned} \tag{10}$$

Now for this problem, $r^2+a^2 = (z\pm b/2)^2+\rho^2 = d^2+z^2+\rho^2\pm bz$. In the first term we can ex pand $(r^2+a^2)^{-3/2} \approx d^{-3}(1-(3/2)(z^2+\rho^2\pm bz)/d^2 + (15/8)b^2z^2/d^4)$ and in the second term we can put $(r^2+a^2)^{-7/2} \approx d^{-7}$. Then

$$\begin{aligned}
\boldsymbol{A}_\pm &= \frac{Ia^2\mu_0}{4\pi d^3}(-\hat{x}y+\hat{y}x)\left[1 - \frac{3}{2}\frac{z^2+\rho^2\pm bz}{d^2} + \frac{15}{8}\frac{a^2\rho^2+b^2z^2}{d^4} + \cdots\right] \\
\boldsymbol{A}_+ + \boldsymbol{A}_- &= \frac{Ia^2\mu_0}{4\pi d^3}(-\hat{x}y+\hat{y}x)\left[2 - 3\frac{z^2+\rho^2}{d^2} + \frac{15}{4}\frac{a^2\rho^2+b^2z^2}{d^4} + \cdots\right] \\
&= \frac{Ia^2\mu_0}{4\pi d^3}(-\hat{x}y+\hat{y}x)\left[2 - 3(a^2-b^2)\frac{z^2-\rho^2/4}{d^4}\right] \equiv \frac{Ia^2\mu_0}{4\pi d^3}(-\hat{x}y+\hat{y}x)f(\rho,z) \\
B^z &= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \frac{Ia^2\mu_0}{4\pi d^3}\left(2f + \rho\frac{\partial f}{\partial \rho}\right) = \frac{Ia^2\mu_0}{4\pi d^3}\left[4 - 6(a^2-b^2)\frac{z^2-\rho^2/2}{d^4}\right] \\
B^x &= -\frac{\partial A_y}{\partial z} = -x\frac{\partial f}{\partial z} = xz\frac{Ia^2\mu_0}{4\pi d^3}6(a^2-b^2)\frac{1}{d^4}, \qquad B^\rho = \rho z\frac{Ia^2\mu_0}{4\pi d^3}6(a^2-b^2)\frac{1}{d^4}
\end{aligned}$$

d) The expansion in b) was set up by writing

$$\begin{aligned}
[(z\pm b/2)^2+a^2]^{-3/2} &= [z^2+d^2\pm bz]^{-3/2} = d^{-3}[1+z^2/d^2\pm bz/d^2]^{-3/2} \\
&= d^{-3}[1+(a^2+b^2/4)z^2/d^4 \pm bz/d^2]^{-3/2}
\end{aligned} \tag{11}$$

which organizes the expansion as a power series in z/d^2. on the other hand an expansion at large $|z|$ requires rewriting

$$[z^2+d^2\pm bz]^{-3/2} = |z|^{-3}[1+d^2/z^2\pm b/z]^{-3/2} = |z|^{-3}[1+(a^2+b^2/4)/z^2\pm b/z]^{-3/2}$$

which would be obtained from the first rewrite by the formal substitution $d \to |z|$.

e) For $b = a$ the result of part b) reduces to

$$B_z \quad \to \quad \frac{\mu_0 I a^2}{d^3}\left(1 - \frac{144 z^4}{125 a^4}\right) \approx \frac{\mu_0 I a^2}{d^3}\left(1 - \left(1.036\frac{|z|}{a}\right)^4\right) \tag{12}$$

For the second term to be less than 10^{-4}, $|z|/a < 1/10.36 \approx .0965$. To be less than 10^{-2} the condition is $|z|/a < 1/\sqrt{10.36} \approx .311$.