Electromagnetic Theory I

Solution Set 9

Due: 13 November 2020

33. J, Problem 5.3

Solution: We regard the solenoid as a stack of current loops. For a current loop at $z = z'$ the field on the z-axis is simply

$$B_z = \frac{\mu_0 I a^2}{2(a^2 + (z - z')^2)^{3/2}} \tag{1}$$

With N loops per unit length, the current in dz' is $NIdz'$, so superposition gives for the total field

$$B_z = N\int_{-L/2}^{L/2} dz' \frac{\mu_0 I a^2}{2(a^2 + (z - z')^2)^{3/2}} = \frac{\mu_0 IN}{2}\left[\sin\left(\tan^{-1}\frac{L-2z}{2a}\right) + \sin\left(\tan^{-1}\frac{L+2z}{2a}\right)\right]$$

From the geometry of the figure in the problem we find

$$\begin{aligned}
\cos\theta_2 &= \frac{L-2z}{\sqrt{4a^2 + (L-2z)^2}} = \sin\tan^{-1}\frac{L-2z}{2a} \\
\cos\theta_1 &= \frac{L+2z}{\sqrt{4a^2 + (L+2z)^2}} = \sin\tan^{-1}\frac{L+2z}{2a} \\
B_z &= \frac{\mu_0 IN}{2}(\cos\theta_1 + \cos\theta_2) \qquad (2)
\end{aligned}$$

34. A point magnetic dipole with moment $\boldsymbol{m} = m\hat{z}$ is placed at the center of a spherical shell with uniform magnetic permeability μ, and with inner and outer radii a, b respectively.

a) Set up the boundary equations that determine the magnetic field in the three regions $0 < r < a$, $a < r < b$, $r > b$.

Solution: Since $\boldsymbol{J} = 0$, we can write $\boldsymbol{H} = -\nabla\phi$. the potential of a dipole is pure $l = 1$, so only need the $l = 1$ part of the potential in all three regions

$$\phi = \frac{1}{4\pi}\begin{cases} \left(\frac{m}{r^2} + A_1 r\right)\cos\theta & 0 < r < a \\ \left(\frac{B_2}{r^2} + A_2 r\right)\cos\theta & a < r < b \\ \frac{B_3}{r^2}\cos\theta & b < r < \infty \end{cases} \tag{3}$$

Continuity of $\boldsymbol{H}_t$ reduces to continuity of ϕ:

$$A_2 b + \frac{B_2}{b^2} = \frac{B_3}{b^2}, \qquad A_2 a + \frac{B_2}{a^2} = A_1 a + \frac{m}{a^2} \tag{4}$$

Continuity of $\boldsymbol{B}_n$ means continuity of $\mu\partial\phi/\partial r$.

$$\mu A_2 - 2\mu\frac{B_2}{b^3} = -2\mu_0\frac{B_3}{b^3}, \qquad \mu A_2 - 2\mu\frac{B_2}{a^3} = \mu_0 A_1 - 2\mu_0\frac{m}{a^3} \tag{5}$$

b) Solve the equations of part a) to find the $\boldsymbol{B}$ and $\boldsymbol{H}$ fields in all three regions.

Solution: The equations at the $r = b$ interface easily give

$$B_2 = \frac{\mu + 2\mu_0}{3\mu}B_3, \qquad A_2 = \frac{2(\mu - \mu_0)}{3\mu}B_3 \tag{6}$$

Plugging these into the $r = a$ interface equations then leads after a few lines of algebra to

$$\begin{aligned}
A_1 &= \frac{2m}{a^3}\frac{(\mu-\mu_0)(a^3-b^3)(\mu+2\mu_0)}{(\mu+2\mu_0)(\mu_0+2\mu)b^3 - 2a^3(\mu-\mu_0)^2} \\
B_3 &= \frac{9m\mu\mu_0 b^3}{(\mu+2\mu_0)(\mu_0+2\mu)b^3 - 2a^3(\mu-\mu_0)^2} \\
A_2 &= \frac{6m\mu_0(\mu-\mu_0)}{(\mu+2\mu_0)(\mu_0+2\mu)b^3 - 2a^3(\mu-\mu_0)^2} \\
B_2 &= \frac{3m\mu_0(\mu+2\mu_0)b^3}{(\mu+2\mu_0)(\mu_0+2\mu)b^3 - 2a^3(\mu-\mu_0)^2}
\end{aligned}$$

The fields are

$$\boldsymbol{H} = -\nabla\phi = \frac{1}{4\pi}\begin{cases} -A_1\hat{z} + \frac{3m\boldsymbol{r}z - mr^2\hat{z}}{r^5} & 0 < r < a \\ -A_2\hat{z} + B_2\frac{3\boldsymbol{r}z - r^2\hat{z}}{r^5} & a < r < b \\ B_3\frac{3\boldsymbol{r}z - r^2\hat{z}}{r^5} & b < r < \infty \end{cases} \tag{7}$$

and of course $\boldsymbol{B} = \mu\boldsymbol{H}$ for $a < r < b$ and $= \mu_0\boldsymbol{H}$ in the inner and outer regions.

c) Discuss the two extreme limits $\mu \to \infty$ and $\mu \to 0$. Determine the limiting $\boldsymbol{B}$ and $\boldsymbol{H}$ fields in each region, and discuss the qualitative differences of the two limits.

Solution: first for $\mu \to \infty$, $\boldsymbol{H} \to 0$ in and outside the shell ($r > a$). this is in accord with the electrostatic analogy of an $\boldsymbol{E}$ field and a conductor. $\boldsymbol{B} = 0$ outside the conductor, but stays finite within the shell

$$\boldsymbol{B} \to \frac{1}{4\pi}\frac{3\mu_0 m}{b^3 - a^3}\left[-\hat{z} + \frac{b^3}{2}\frac{3\boldsymbol{r}z - r^2\hat{z}}{r^5}\right], \qquad a < r < b \tag{8}$$

In the inner region $\boldsymbol{B} = \mu_0\boldsymbol{H}$ with

$$\boldsymbol{H} \to \frac{m}{4\pi}\left[-\frac{r^3 - a^3}{a^3 r^3}\hat{z} + \frac{3\boldsymbol{r}z}{r^5}\right], \qquad 0 < r < a \tag{9}$$

We see that $\boldsymbol{H}$ is normal to the interface at $r = a$ again in accord with the electrostatic analogy.

In the opposite limit $\mu \to 0$ $\boldsymbol{B} = 0$ within and outside the sphere in accord with the behavior of a superconductor. This time $\boldsymbol{H}$ is nonzero within the shell

$$\boldsymbol{H} \quad \to \quad \frac{1}{4\pi}\frac{3m}{b^3 - a^3}\left[\hat{z} + b^3\frac{3\boldsymbol{r}z - r^2\hat{z}}{r^5}\right], \qquad a < r < b \tag{10}$$

In the inner region we have

$$\boldsymbol{H} \quad \to \quad \frac{m}{4\pi}\left[-\frac{2r^3 + a^3}{a^3 r^3}\hat{z} + \frac{3\boldsymbol{r}z}{r^5}\right], \qquad 0 < r < a \tag{11}$$

At $r = a$ this $\boldsymbol{H}$ is tangential to the interface again in accord with superconducting behavior.

d) Calculate the difference $\Delta U = U_{\mu=0} - U_{\mu=\infty}$ of the total magnetic field energy stored in the two limiting situations. Handle the divergence when $r \to 0$ by excluding the region $0 < r \le \delta \ll a$ in the energy integral. The divergence cancels in ΔU, and you can then take $\delta \to 0$. Can you understand the sign in terms of the qualitative behavior of the fields?

Solution: The field energy density is $\boldsymbol{H} \cdot \boldsymbol{B}/2$. In both of the limits, either $\boldsymbol{B} = 0$ or $\boldsymbol{H} = 0$ (or both) inside and outside the shell. So we only integrate over the region $\delta < r < a$ which we call $\mathcal{R}_\delta$.

$$\begin{aligned} U &= \frac{\mu_0}{2}\int_{\mathcal{R}_\delta} d^3x \nabla\phi \cdot \nabla\phi = \frac{\mu_0}{2}\oint_{\partial\mathcal{R}_\delta} d^2S\hat{n} \cdot \phi\nabla\phi \\ &= \frac{\mu_0}{2}\int d\Omega\left(a^2\phi\frac{\partial\phi}{\partial r}\bigg|_{r=a} - \delta^2\phi\frac{\partial\phi}{\partial r}\bigg|_{r=\delta}\right) = -\frac{\mu_0}{2}\int d\Omega\delta^2\phi\frac{\partial\phi}{\partial r}\bigg|_{r=\delta} \end{aligned} \tag{12}$$

after an integration by parts using $\nabla^2\phi = 0$ throughout $\mathcal{R}_\delta$ and the fact that either $\phi = 0$ or $\partial\phi/\partial r = 0$ at $r = a$. Now

$$\begin{aligned} \delta^2\phi\frac{\partial\phi}{\partial r}\bigg|_{r=\delta} &= \frac{\cos^2\theta}{(4\pi)^2}\delta^2\left(\frac{m}{\delta^2} + A_1\delta\right)\left(-\frac{2m}{\delta^3} + A_1\right) = \frac{\cos^2\theta}{(4\pi)^2}\left(-\frac{2m^2}{\delta^3} - mA_1 + A_1^2\delta^3\right) \\ \int d\Omega\cos^2\theta &= 2\pi\int_{-1}^{1} dx x^2 = \frac{4\pi}{3}, \qquad U = \frac{\mu_0}{24\pi}\left(\frac{2m^2}{\delta^3} + mA_1 - A_1^2\delta^3\right) \\ \Delta U &\to \frac{m\mu_0}{24\pi}(A_1(\mu = 0) - A_1(\mu = \infty)) = \frac{m^2\mu_0}{8\pi a^3} \end{aligned} \tag{13}$$

The positive energy difference reflects the fact that the field lines are denser when they are required to be parallel to the interface than when they are allowed to be perpendicular to the interface.

35. The quantitative description of field lines is rarely discussed. In this problem we find the field line curves for the magnetic field produced by a spherical ball of radius b with permeability μ immersed in a uniform magnetic field $\boldsymbol{H}_0 = H_0\hat{z}$. From the $a \to 0$ limit of the spherical shell problem discussed in class, we know that the exterior field has the form

$$\boldsymbol{H} = H_0\hat{z} + m\frac{3z\boldsymbol{r} - r^2\hat{z}}{4\pi r^5}, \qquad m = \frac{4\pi b^3(\mu-\mu_0)H_0}{\mu+2\mu_0}, \qquad r > b \tag{14}$$

Describing the field line curve parametrically by $\boldsymbol{r}(\lambda)$, the field line is defined by the requirement that its tangent $d\boldsymbol{r}/d\lambda$ at the point $\boldsymbol{r}(\lambda)$ be parallel to the field $\boldsymbol{H}(\boldsymbol{r}(\lambda))$ at that point. Consider the field lines of the above field in the xz plane.

a) In spherical coordinates, the curve can be specified by giving the radius as a function of the polar angle $r(\theta)$. Find the first order differential equation satisfied by $r(\theta)$.

Solution: For a line in the xz-plane, we describe it by $x(\theta) = r(\theta)\sin\theta, z(\theta) = r(\theta)\cos\theta$. Then $d\boldsymbol{r}/d\theta$ will be parallel to $\boldsymbol{H}(\boldsymbol{r}(\theta)$ if

$$\begin{aligned}\frac{\sin\theta dr/d\theta + r\cos\theta}{\cos\theta dr/d\theta - r\sin\theta} &= \frac{H_x}{H_z} = \frac{3m\sin\theta\cos\theta}{H_0 4\pi r^3 + m(3\cos^2\theta - 1)}\\ \frac{1}{r}\frac{dr}{d\theta} &= -\frac{4\pi H_0 r^3 + 2m}{4\pi H_0 r^3 - m}\cot\theta\end{aligned}$$

b) Find the general solution of this equation. The values of a single integration constant will distinguish different field lines.

Solution:

$$\begin{aligned}\frac{dr}{r}\frac{4\pi H_0 r^3 - m}{4\pi H_0 r^3 + 2m} &= \frac{1}{2}d\ln\frac{4\pi H_0 r^3 + 2m}{r} = -d\ln\sin\theta\\ 4\pi H_0 + \frac{2m}{(z^2+x^2)^{3/2}} &= \frac{K}{r^2\sin^2\theta} = \frac{K}{x^2} = \frac{4\pi H_0 x_\infty^2}{x^2}\\ (z^2+x^2)^{3/2} &= \frac{2mx^2}{4\pi H_0(x_\infty^2 - x^2)} = \frac{2b^3(\mu-\mu_0)x^2}{(\mu+2\mu_0)(x_\infty^2 - x^2)}\end{aligned} \tag{15}$$

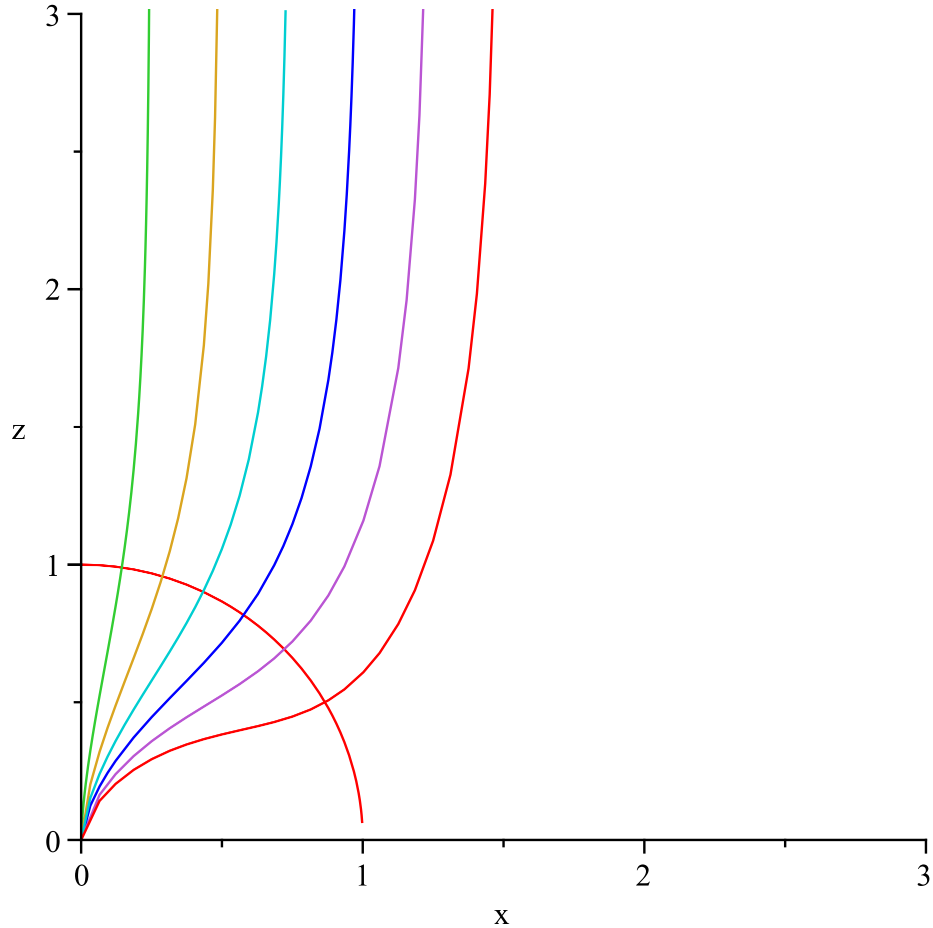
From which we obtain $z(x)$. In this formula x_∞ is the asymptotic value of x as $z \to \infty$. Notice that if $\mu > \mu_0$, $x^2 < x_\infty^2$ and if $\mu < \mu_0$, $x^2 > x_\infty^2$. Of course for $\mu = \mu_0$ the equation is consistent only if $x = x_\infty$, i.e. if the lines are linear and vertical.

c) Plot a typical set of field lines for the two cases $\mu = 0$ and $\mu = \infty$. Choose them to be equally spaced in x as they approach $z = \infty$. Compare and contrast the resulting pictures in the two cases. You may restrict the field lines to the quadrant $x > 0, z > 0$.

Solution: For purposes of plotting we set $b = 1$ and $\xi = 2(\mu-\mu_0)/(\mu+2\mu_0)$. Then we have

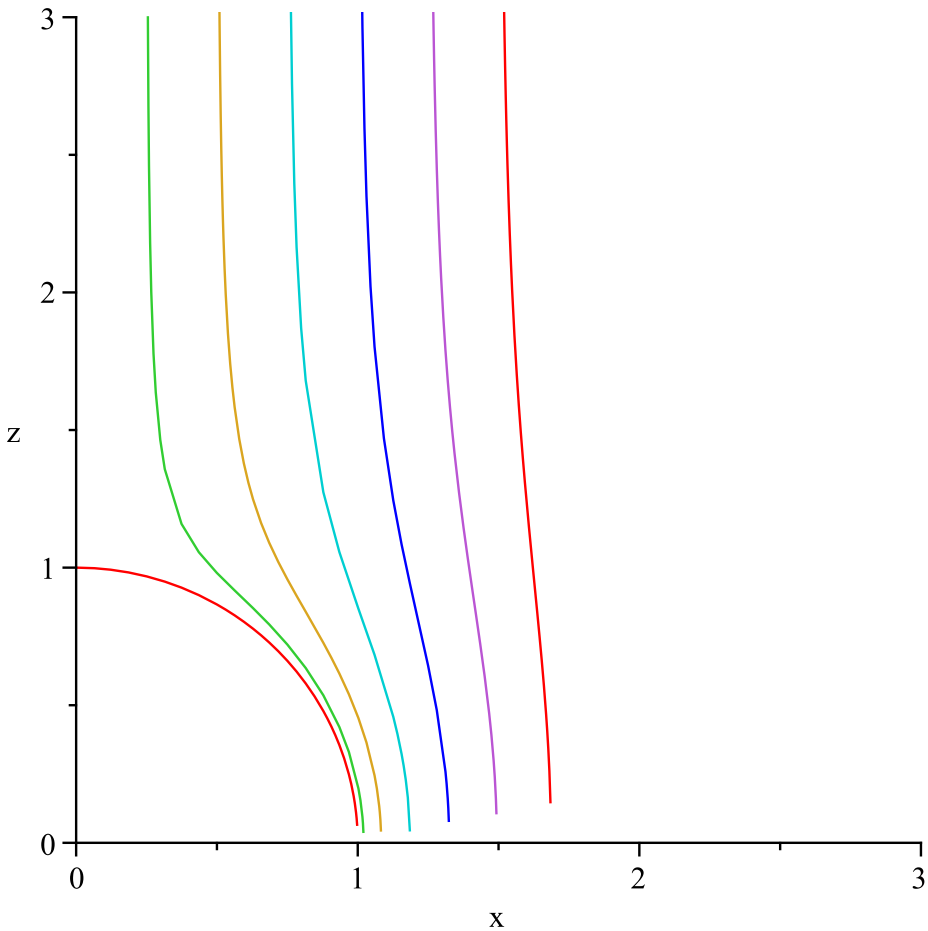
$$z(x) = \sqrt{\left(\frac{\xi x^2}{x_\infty^2 - x^2}\right)^{2/3} - x^2} \tag{16}$$

We plot the field lines for the two cases $\mu = \infty$, $(\xi = 2)$



In the above figure, disregard the field lines inside the sphere. The $\boldsymbol{H}$ lines should terminate at the surface of the sphere, since $\boldsymbol{H} = 0$ for $r < 1$. If the lines are thought of as $\boldsymbol{B}$ lines, $\boldsymbol{B}$ is uniform inside the sphere so its field lines should be vertical and equally spaced for $r < 1$: drop a vertical from the intersection of each line with the surface $r = 1$ to get the $\boldsymbol{B}$ lines inside.

and for $\mu = 0$ $(\xi = -1)$.



In both cases we have chosen x_∞ values separated by 0.25. We see the characteristic behavior of $\mu = \infty$, namely the field lines are normal to the boundary like a magnetic conductor. And for $\mu = 0$ we see the behavior of a superconductor: field lines are expelled from the material and tangential to the boundary.

36. J, Problem 5.22

Solution: We can replace the uniformly magnetized bar with a surface magnetic charge density at each end given by M at one end and $-M$ at the other end. The material of infinite permeability then behaves like a conductor of magnetic charge, so it is equivalent to an image magnetic charge distribution. When one end of the bar is flush with the material, that end's image is a zero distance from it. The magnetic charges from the distant ends of the original magnet and its image produce negligible fields, so the force is just the force between oppositely charged capacitor plates separated by zero distance. Then the $\boldsymbol{H}$ field between the plates is uniform and of magnitude M. The energy stored between the plates is $BHAd/2 = \mu_0 M^2 Ad/2$ so the magnetude of the force of the image plate on the original bar magnet is $F = \mu_0 M^2 A/2$.