Electromagnetic Theory I

Solution Set 10

Due: 20 November 2020

37. J, Problem 5.17. A shortcut to the solution to this problem is to relate it to an electrostaic analogue discussed in class, a charge distribution above a half space filling dielectric medium. Or you can solve it from scratch.

Solution: Rather than solving this problem from scratch, we instead relate it to the corresponding problem with a charge q above a dielectric solved in class. Recall that the boundary conditions $\boldsymbol{E}_{\rm tan}, \boldsymbol{D}_n$ continuous were satisfied with a field above the dielectric given by the charge q and an image charge $q' = -q(\epsilon - \epsilon_0)/(\epsilon + \epsilon_0)$ the same distance below the dielectric boundary. The field below the dielectric boundary is given by that of a charge $q'' = 2q\epsilon/(\epsilon + \epsilon_0)$. Applying this to a point electric dipole above the dielectric shows that the image dipole moments are $p'_z = p_z(\epsilon - \epsilon_0)/(\epsilon + \epsilon_0)$, $p'_{x,y} = -p_{x,y}(\epsilon - \epsilon_0)/(\epsilon + \epsilon_0)$ and $\boldsymbol{p}'' = \boldsymbol{p}2\epsilon/(\epsilon + \epsilon_0)$.

a) Now we make the analogy. A magnetic dipole above a permeable material produces the magnetic field, above the material, of the dipole plus an image dipole $m'_z = m_z(\mu - \mu_0)/(\mu + \mu_0)$ the same distance below the material boundary. If we have instead a dipole density $\boldsymbol{M}$ above the material, the image distribution is accordingly given by

$$M'_z(x,y,z) = M_z(x,y,-z)\frac{\mu_r - 1}{\mu_r + 1}, \qquad M'_{x,y}(x,y,z) = -M_{x,y}(x,y,-z)\frac{\mu_r - 1}{\mu_r + 1}$$

The corresponding image current is $\boldsymbol{J}' \equiv \nabla \times \boldsymbol{M}'$:

$$\begin{aligned} J'_z(x,y,z) &= \partial_x M'_y - \partial_y M'_x = -J_z\frac{\mu_r - 1}{\mu_r + 1}(x,y,-z) \\ J'_x(x,y,z) &= \partial_y M'_z - \partial_z M'_y = +J_x\frac{\mu_r - 1}{\mu_r + 1}(x,y,-z) \\ J'_y(x,y,z) &= \partial_z M'_x - \partial_x M'_z = +J_y\frac{\mu_r - 1}{\mu_r + 1}(x,y,-z) \end{aligned}$$

as desired.

b) The field below the material boundary is given by the image magnetization $\boldsymbol{M}'' = 2\mu_2/(\mu_r + 1)\boldsymbol{M}$ which corresponds to the current distribution

$$\boldsymbol{J}'' = \nabla \times \boldsymbol{M}'' = \boldsymbol{J}\frac{2\mu_r}{\mu_r + 1} \qquad (1)$$

as desired.

38. In studying a quantum particle in the presence of an electromagnetic magnetic field we used the Lagrangian

$$L \;=\; \frac{1}{2}m\dot{\boldsymbol{r}}^2 + q\dot{\boldsymbol{r}}\cdot\boldsymbol{A}(\boldsymbol{r},t) - q\phi(\boldsymbol{r},t) \tag{2}$$

When there is time dependence we showed near the beginning of the course that the relation between fields and potentials is

$$\boldsymbol{B}(\boldsymbol{r},t) = \nabla\times\boldsymbol{A}(\boldsymbol{r},t), \qquad \boldsymbol{E}(\boldsymbol{r},t) = -\nabla\phi(\boldsymbol{r},t) - \frac{\partial\boldsymbol{A}}{\partial t}(\boldsymbol{r},t) \tag{3}$$

a) Show that Lagrange's equations of motion with this Lagrangian imply Newton's equation with the Lorentz force on the right side.

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v}\times\boldsymbol{B} \tag{4}$$

Solution: Lagrange's equations are

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial\dot{\boldsymbol{r}}} &= \frac{\partial L}{\partial\boldsymbol{r}}, \qquad \frac{\partial L}{\partial\dot{\boldsymbol{r}}} = m\dot{\boldsymbol{r}} + q\boldsymbol{A}, \qquad \frac{\partial L}{\partial\boldsymbol{r}} = q\dot{r}^i\nabla A_i - q\nabla\phi \\
& m\ddot{\boldsymbol{r}} + q\dot{\boldsymbol{r}}\cdot\nabla\boldsymbol{A} + q\frac{\partial\boldsymbol{A}}{\partial t} = q\dot{r}^i\nabla A_i - q\nabla\phi \\
& m\ddot{\boldsymbol{r}} = -q\dot{\boldsymbol{r}}\cdot\nabla\boldsymbol{A} + q\dot{r}^i\nabla A_i - q\nabla\phi - q\frac{\partial\boldsymbol{A}}{\partial t} = q(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})
\end{aligned} \tag{5}$$

b) Derive the Hamiltonian for this system.

Solution:

$$H \;\equiv\; \boldsymbol{p}\cdot\dot{\boldsymbol{r}} - L = m\dot{\boldsymbol{r}}^2 + q\dot{\boldsymbol{r}}\cdot\boldsymbol{A} - L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 + q\phi = \frac{(\boldsymbol{p}-q\boldsymbol{A})^2}{2m} + q\phi \tag{6}$$

c) Repeat the discussion of parts a) and b) for the relativistic Lagrangian obtained by replacing the nonrelativistic kinetic energy term in L as follows

$$\frac{1}{2}m\dot{\boldsymbol{r}}^2 \to mc^2\left(1-\sqrt{1-\frac{\dot{\boldsymbol{r}}^2}{c^2}}\right) \tag{7}$$

Solution: The only change is

$$\begin{aligned}
\frac{\partial L}{\partial\dot{\boldsymbol{r}}} &\to \frac{m\dot{\boldsymbol{r}}}{\sqrt{1-\dot{\boldsymbol{r}}^2/c^2}} + q\boldsymbol{A}, \qquad \frac{d}{dt}\frac{m\dot{\boldsymbol{r}}}{\sqrt{1-\dot{\boldsymbol{r}}^2/c^2}} = q(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}) \\
H &= \frac{m\dot{\boldsymbol{r}}^2}{\sqrt{1-\dot{\boldsymbol{r}}^2/c^2}} + mc^2\sqrt{1-\dot{\boldsymbol{r}}^2/c^2} - mc^2 + q\phi = \frac{mc^2}{\sqrt{1-\dot{\boldsymbol{r}}^2/c^2}} - mc^2 + q\phi
\end{aligned}$$

Solving for

$$\begin{aligned}
\dot{\boldsymbol{r}}^2 &= \frac{(\boldsymbol{p}-q\boldsymbol{A})^2c^2}{mc^2+(\boldsymbol{p}-q\boldsymbol{A})^2}, \qquad \frac{mc^2}{\sqrt{1-\dot{\boldsymbol{r}}^2/c^2}} = \sqrt{m^2c^4+(\boldsymbol{p}-q\boldsymbol{A})^2c^2} \\
H &= \sqrt{m^2c^4+(\boldsymbol{p}-q\boldsymbol{A})^2c^2}-mc^2+q\phi \qquad (8)
\end{aligned}$$

39. J, Problem 5.25.

Solution:

a) For the straight wire, $\boldsymbol{B} = I_2\mu_0\hat{\phi}/2\pi\rho$ with $\rho = \sqrt{x^2+y^2}$. Then we can choose $\boldsymbol{A} = -(I_2\hat{z}\ln\rho^2)/4\pi$. Then

$$W_{12} = I_1\oint d\boldsymbol{l}\cdot\boldsymbol{A} = -\frac{I_1I_2\mu_0}{4\pi}\oint d\boldsymbol{l}\cdot\hat{z}\ln\rho^2 = \frac{aI_1I_2\mu_0}{4\pi}\ln\frac{\rho_2^2}{\rho_1^2} \qquad (9)$$

where $\rho_1^2 = d^2+\frac{b^2}{4}-bd\cos\alpha$ and $\rho_2^2 = d^2+\frac{b^2}{4}+bd\cos\alpha$.

$$W_{12} = \frac{aI_1I_2\mu_0}{4\pi}\ln\frac{4d^2+b^2+4bd\cos\alpha}{4d^2+b^2-4bd\cos\alpha} \qquad (10)$$

b) In accord with our discussion of force from energy at constant currents, we have

$$F = +\frac{\partial W_{12}}{\partial d} = -4ab\frac{I_1I_2\mu_0}{\pi}\cos\alpha\frac{2d^2-b^2}{(4d^2+b^2)^2-16b^2d^2\cos^2\alpha} \qquad (11)$$

Since it is negative when $\cos\alpha > 0$, the force is toward the wire when the current parallel to the wire is nearest the wire.

c) Here it is easiest to calculate the flux Φ_2 of the magnetic field of the straight wire through the loop and use $W_{12} = I_1\Phi_2$.

$$\begin{aligned}
\Phi_2 &= \int dS\hat{n}\cdot\hat{\varphi}\frac{I_2\mu_0}{2\pi\rho} = \int_{-a}^{a}dz\int_{\rho_1(z)}^{\rho_2(z)}d\rho\frac{I_2\mu_0}{2\pi\rho} = \frac{I_2\mu_0}{4\pi}\int_{-a}^{a}dz\ln\frac{\rho_2^2(z)}{\rho_1^2(z)} \\
\rho_2^2(z) &= d^2+a^2-z^2+2d\sqrt{a^2-z^2}\cos\alpha = (d+e^{i\alpha}\sqrt{a^2-z^2})(d+e^{-i\alpha}\sqrt{a^2-z^2}) \\
\rho_1^2(z) &= (d-e^{i\alpha}\sqrt{a^2-z^2})(d-e^{-i\alpha}\sqrt{a^2-z^2}) \qquad (12)
\end{aligned}$$

Note that the factor $\hat{n}\cdot\hat{\varphi}$ just projects the area element dS onto the plane perpendicular to $\hat{\varphi}$ so that $dS\hat{n}\cdot\hat{\varphi} = dzd\rho$. At a fixed z the limits on the ρ integral are just given by $\rho_1(z) < \rho < \rho_2(z)$. $\rho_{1,2}$ are just the radial coordinates of the circular loop at that value of z. We can do the integral by first making the trig substitution $z = a\sin\theta$

$$\Phi_2 = \frac{I_2\mu_0a}{4\pi}\int_{-\pi/2}^{\pi/2}d\theta\cos\theta\ln\frac{(d+ae^{i\alpha}\cos\theta)(d+ae^{-i\alpha}\cos\theta)}{(d-ae^{i\alpha}\cos\theta)(d-ae^{-i\alpha}\cos\theta)} \qquad (13)$$

Now write $\cos\theta = d\sin\theta/d\theta$ and integrate by parts to get rid of the ln's:

$$\Phi_2 = 2\frac{I_2\mu_0 a}{4\pi}\mathrm{Re}\int_{-\pi/2}^{\pi/2} d\theta \frac{2ade^{i\alpha}\sin^2\theta}{d^2 - a^2e^{2i\alpha}\cos^2\theta} = \frac{I_2\mu_0 a}{4\pi}\mathrm{Re}\oint \frac{dz}{iz}\frac{2ade^{i\alpha}(2z - z^2 - 1)}{4d^2z - a^2e^{2i\alpha}(2z + z^2 + 1)}$$

where we changed variables to $z = e^{2i\theta}$. The integral can be done by residues. The pole locations are $z = 0, z_\pm$ where

$$z_\pm = 2\frac{d^2}{a^2}e^{-2i\alpha} - 1 \pm 2\frac{d^2}{a^2}e^{-i\alpha}\sqrt{e^{-2i\alpha} - \frac{a^2}{d^2}} \tag{14}$$

The contour on the unit circle encloses the poles at $z = 0, z_-$, so evaluating their residues we obtain

$$W_{12} = I_1I_2\mu_0 d\mathrm{Re}\left[e^{-i\alpha} - \sqrt{e^{-2i\alpha} - \frac{a^2}{d^2}}\right] = I_1I_2\mu_0\mathrm{Re}\left[de^{i\alpha} - \sqrt{e^{2i\alpha}d^2 - a^2}\right]$$

$$F = \frac{\partial W_{12}}{\partial d} = I_1I_2\mu_0\mathrm{Re}\left[e^{i\alpha} - \frac{de^{2i\alpha}}{\sqrt{e^{2i\alpha}d^2 - a^2}}\right] \tag{15}$$

d)

$$W_{12}^a \sim \frac{aI_1I_2\mu_0}{4\pi}\ln\left[1 + 2\frac{b}{d}\cos\alpha\right] \sim \frac{abI_1I_2\mu_0}{2d\pi}\cos\alpha = mB\cos\alpha = \boldsymbol{m}\cdot\boldsymbol{B} \tag{16}$$

since α is the angle between the normal to the loop and $\hat{\phi}$ which is the direction of $\boldsymbol{B}$.

$$W_{12}^c \sim I_1I_2\mu_0\mathrm{Re}\left[de^{i\alpha}(1 - (1 - a^2e^{-2i\alpha}/2d^2))\right] = I_1I_2\mu_0\mathrm{Re}\left[de^{i\alpha}a^2e^{-2i\alpha}/2d^2\right]$$
$$\sim \frac{\pi I_1I_2a^2\mu_0}{2\pi d}\cos\alpha = mB\cos\alpha = \boldsymbol{m}\cdot\boldsymbol{B} \tag{17}$$

The energy of a permanent dipole is $-\boldsymbol{m}\cdot\boldsymbol{B}$. The opposite sign comes from the fact that the above energies are calculated at fixed currents not fixed potential.

40. J, Problem 5.32

Solution:

a) The finite thickness wire can be regarded as a superposition of loops. Consider the contribution of a loop at ρ', φ' which has a radius $a + \rho'\cos\varphi'$. In the formula for k^2 this radius will be used for a. The quantity $r\sin\theta$ will be replaced by $a + \rho\cos\varphi$, and $r \to \sqrt{(\rho\sin\varphi - \rho'\sin\varphi')^2 + (a + \rho\cos\varphi)^2}$. since $\rho, \rho' << a$, all these quantities are close to a and k^2 is close to 1. Thus (5.37) can be approximated

$$\begin{aligned}
A_\varphi &\approx \frac{\mu_0 I}{2\pi}\left[\ln\frac{4}{\sqrt{1-k^2}} - 2\right] \\
1-k^2 &= \frac{a^2+r^2-2ar\sin\theta}{a^2+r^2+2ar\sin\theta} \\
&\to \frac{(a+\rho'\cos\varphi')^2+(\rho\sin\varphi-\rho'\sin\varphi')^2+(a+\rho\cos\varphi)^2-2(a+\rho'\cos\varphi')(a+\rho\cos\varphi)}{4a^2} \\
&\to \frac{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')}{4a^2} = \frac{(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}{4a^2} \qquad (18)
\end{aligned}$$

where $\boldsymbol{\rho}, \boldsymbol{\rho}'$ are two dimensional vectors. Then the contribution of the bundle of current loops contributes

$$A_\varphi \approx \int_{\rho'<b} \frac{\mu_0 d^2\rho'}{2\pi}\frac{I}{\pi b^2}\left[\ln\frac{8a}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|} - 2\right] \qquad (19)$$

This is the potential of a uniform cylindrical current distribution and since $\boldsymbol{\rho}$ lies outside the distribution, it's the same as the potential with all the current at $\rho' = 0$:

$$A_\varphi \approx \frac{\mu_0 I}{2\pi}\left[\ln\frac{8a}{\rho} - 2\right] \qquad (20)$$

b)Since the current is uniform inside the wire, Ampere's Law shows that the magnetic field within the wire is

$$B = \frac{\mu_0 I \rho}{2\pi b^2} \qquad (21)$$

with field lines in circles about the wire. This corresponds to a vector potential

$$A_\varphi = -\frac{\mu_0 I \rho^2}{4\pi b^2} + C, \qquad \rho < b \qquad (22)$$

Picking the constant C so that A_φ is continuous with our result for $\rho > b$ at $\rho = b$ yields

$$A_\varphi = \frac{\mu_0 I}{4\pi}\left[1 - \frac{\rho^2}{b^2}\right] + \frac{\mu_0 I}{2\pi}\left[\ln\frac{8a}{b} - 2\right], \qquad \rho < b \qquad (23)$$

c) Finally we calculate the field energy

$$\begin{aligned}
U &= \frac{1}{2}\int d^3x \boldsymbol{A}\cdot\boldsymbol{J} = \frac{2\pi a}{2}\int_0^b \rho d\rho \int_0^{2\pi} d\phi \frac{I}{\pi b^2} A_\varphi \\
&= \frac{I^2 a\mu_0}{2b^2}\int_0^b d\rho\left[\rho - \frac{\rho^3}{b^2} + 2\rho\left(\ln\frac{8a}{b} - 2\right)\right] = \frac{I^2 a\mu_0}{2}\left[\frac{1}{2} - \frac{1}{4} + \ln\frac{8a}{b} - 2\right] \\
&= \frac{I^2 a\mu_0}{2}\left[\ln\frac{8a}{b} - \frac{7}{4}\right] \\
L &= \mu_0 a\left[\ln\frac{8a}{b} - \frac{7}{4}\right] \qquad (24)
\end{aligned}$$

If the current is concentrated at the surface of the wire, the energy integral is simply

$$U = \pi I a A_\varphi(\rho = b) = \frac{\mu_0 a I^2}{2}\left[\ln\frac{8a}{b} - 2\right], \qquad L = \mu_0 a\left[\ln\frac{8a}{b} - 2\right] \tag{25}$$

Only the constant term is modified. The ln term is universal.