Electromagnetic Theory I

Solution Set 11

Due: 2 December 2020

41. Motional Emf from a Spinning Magnet (Revision of J, 6.4). Let the magnet be a uniformly magnetized sphere of radius R and magnetization $\boldsymbol{M} = M\hat{z}$, which is also a conductor with finite resistivity. Set the magnet spinning with angular speed ω about the magnetization axis. Assume the system reaches a steady state so that no current flows in the conductor's rest frame. Also assume that there is no net charge on the magnet.

a) Any charge density at rest in the magnet's rest frame must arrange itself so that the force from its electric field cancels the magnetic force from its motion in the magnet's magnetic field. Find this canceling electric field, and from it show that the steady state charge density in the bulk is $\rho = -4M\omega/3c^2 = -m\omega/\pi c^2 R^3$, where m is the total magnetic moment.

Solution: A charge moving with the magnet has a velocity $\boldsymbol{\omega} \times \boldsymbol{r} = \omega r_\perp \hat{\phi}$. The magnetic field inside the sphere from the uniform magnetization is $2\mu_0 \boldsymbol{M}/3$, and exerts a force $2q\mu_0\omega M \boldsymbol{r}_\perp/3$. Equilibrium requires an electric field $\boldsymbol{E} = -2\mu_0\omega M\boldsymbol{r}_\perp/3$ to balance this force. This will be supplied by a charge density $\rho = \epsilon_0 \nabla \cdot \boldsymbol{E} = -4\mu_0\epsilon_0\omega M/3 = -4\omega M/3c^2 = -m\omega/\pi c^2 R^3$, where m is the total magnetic moment. [Strictly speaking the electric force shouldn't exactly cancel the magnetic force, but combine with it to yield a net centripetal force to keep the charge moving in a circle. That centripetal force must be $-m_q \boldsymbol{v} \times \boldsymbol{\omega} = -m_q\omega^2 \boldsymbol{r}_\perp$ where m_q is the mass of the charge q. We neglect this term which means we are assuming $m_q\omega^2 \ll |q|\mu_0\omega M$ or $\omega \ll |q|\mu_0 M/m_q$.]

b) By examining the scalar potential for the electric field found in a) show that it is a superposition of electric monopole and quadrupole potentials. The monopole contribution must be cancelled by charge on the surface.

Solution: $\boldsymbol{E} = -\nabla\phi$, with

$$\phi_{in} = \mu_0\omega M \frac{x^2+y^2}{3} = \frac{\mu_0\omega M}{3} r^2 \sin^2\theta = \frac{2\mu_0\omega M}{9} r^2 (1 - P_2(\cos\theta)) \qquad (1)$$

$r^2 P_2(\cos\theta)$ is coordinate dependence of a quadrupole solution of Laplace's equation regular at the origin.

c) Since there is no net charge on the magnet, the field outside has no monopole contribution. From the boundary conditions at the surface (continuous tangential electric fields), derive the potential outside the sphere. Then from the difference in normal fields outside and in, show that the surface charge density is

$$\sigma(\theta) = \frac{m\omega}{3\pi c^2 R^2}\left(1 - \frac{5}{2}P_2(\cos\theta)\right) \qquad (2)$$

Solution: The exterior quadrupole potential with good behavior at infinity is

$$\phi_{out} = -\frac{2\mu_0\omega M R^5}{9r^3} P_2(\cos\theta) \quad (3)$$

where the strength is determined by continuity of $E_t \propto \partial\phi/\partial\theta$ at $r = R$. By Gauss' Law we then have

$$\sigma = -\epsilon_0 \left[\frac{\partial\phi_{out}}{\partial r} - \frac{\partial\phi_{in}}{\partial r}\right] = \frac{4\epsilon_0\mu_0\omega M R}{9}\left(1 - \frac{5}{2}P_2(\cos\theta)\right) = \frac{\omega m}{3\pi c^2 R^2}\left(1 - \frac{5}{2}P_2(\cos\theta)\right) \quad (4)$$

d) By calculating the line integral of the electric field from a point on the equator to the north pole show that the induced Emf $= \mu_0 m\omega/4\pi R$.

Solution: Choose a radial contour from a point on the equator to the center of the sphere and then up the axis to the north pole. Then since $\boldsymbol{E}$ is $\perp$ to the z-axis, the second segment gives zero. Then

$$\text{Emf} = \frac{2\mu_0\omega M}{3}\int_0^R dr r = \frac{\mu_0\omega M R^2}{3} = \frac{\mu_0 m\omega}{4\pi R} \quad (5)$$

42. J, Problem 6.5.

Solution:

a) We just need an integration by parts:

$$\begin{aligned}\boldsymbol{P}_{\text{field}} &= \int d^3x \boldsymbol{D}\times\boldsymbol{B} = \epsilon\mu\int d^3x(-\nabla\phi)\times\boldsymbol{H} = \frac{1}{c'^2}\int d^3x\phi\nabla\times\boldsymbol{H} - \frac{1}{c'^2}\oint dA\phi\hat{n}\times\boldsymbol{H}\\ &= \frac{1}{c'^2}\int d^3x\phi\boldsymbol{J}\end{aligned} \quad (6)$$

where the surface term is dropped, assuming the fields fall off fast enough at large distances.

b) Expanding $\phi(\boldsymbol{r}) = \phi(0) + \boldsymbol{r}\cdot\nabla\phi(0) + \cdots$, we have

$$\boldsymbol{P}_{\text{field}} = \frac{1}{c'^2}\int d^3x(\phi(0)\boldsymbol{J} - \boldsymbol{J}\boldsymbol{r}\cdot\boldsymbol{E}(0) + \cdots) \quad (7)$$

Next we recall that for constant current density, $\int \boldsymbol{J} = 0$ and $\int r^i J^j = -\int r^j J^i$. Thus the first term is zero and integral in the second term is

$$\begin{aligned}\int d^3x r^i E^i J^j &= \frac{1}{2}\int d^3x E^i(r^i J^j - r^j J^i) = -\frac{1}{2}\int d^3x[\boldsymbol{E}(0)\times(\boldsymbol{r}\times\boldsymbol{J})]^j = -[\boldsymbol{E}(0)\times\boldsymbol{m}]^j\\ \boldsymbol{P}_{\text{field}} &= \frac{1}{c'^2}\boldsymbol{E}(0)\times\boldsymbol{m}\end{aligned} \quad (8)$$

c) For a uniform electric field the surface term from the integration by parts will not vanish, so we don't integrate by parts, and instead work with the original expression

$$\begin{aligned}
\boldsymbol{P}_{\text{field}} &= \int d^3x \boldsymbol{D}\times\boldsymbol{B} = \frac{1}{c'^2}\boldsymbol{E}(0)\times\int d^3x\boldsymbol{H} \\
\boldsymbol{m} &= \frac{1}{2}\int d^3x \boldsymbol{r}\times(\nabla\times\boldsymbol{H}) = \frac{1}{2}\int d^3x(\boldsymbol{r}^i\nabla H^i - r^i\nabla^i\boldsymbol{H}) \\
&= \int d^3x\boldsymbol{H} - \frac{1}{2}\oint dA(\hat{r}\boldsymbol{r}\cdot\boldsymbol{H} - r\boldsymbol{H}) \qquad (9)
\end{aligned}$$

We take the closed surface in the second term to be a sphere of very large radius. At large distances $\boldsymbol{H}\sim(3\hat{r}\hat{r}\cdot\boldsymbol{m}-\boldsymbol{m})/4\pi r^3$ is approximately a dipole field, Then

$$\begin{aligned}
\hat{r}\boldsymbol{r}\cdot\boldsymbol{H} - r\boldsymbol{H} &\sim \frac{\hat{r}2r\hat{r}\cdot\boldsymbol{m} - r((3\hat{r}\hat{r}\cdot\boldsymbol{m}-\boldsymbol{m})}{4\pi r^3} = \frac{-\hat{r}\hat{r}\cdot\boldsymbol{m}+\boldsymbol{m}}{4\pi r^2} \\
\oint dA(\hat{r}\boldsymbol{r}\cdot\boldsymbol{H} - r\boldsymbol{H}) &= -\int d\Omega\frac{\boldsymbol{m}-\hat{r}\hat{r}\cdot\boldsymbol{m}}{4\pi} = -\frac{2}{3}\boldsymbol{m} \qquad (10)
\end{aligned}$$

Then

$$\begin{aligned}
\boldsymbol{m} &= \int d^3x\boldsymbol{H} + \frac{\boldsymbol{m}}{3}, \qquad \text{or} \qquad \int d^3x\boldsymbol{H} = \frac{2}{3}\boldsymbol{m} \\
\boldsymbol{P}_{\text{field}} &= \frac{1}{c'^2}\boldsymbol{E}(0)\times\int d^3x\boldsymbol{H} = \frac{2}{3}\frac{1}{c'^2}\boldsymbol{E}(0)\times\boldsymbol{m}
\end{aligned}$$

as desired. The result for $\int d^3x\boldsymbol{H}$ is equivalent to the result in Jackson's Eq.(5.62).

43. Consider the ideal circular parallel plate capacitor of radius a and plate separation $d \ll a$, hooked up to a straight current-carrying wire on the axis as pictured in the figure to J, Problem 6.14. The current in the wire varies harmonically, $I(t) = I_0\cos\omega t = \text{Re } I_0 e^{-i\omega t}$. In this problem we neglect the effect of fringing fields, which means that the fields within the capacitor are assumed to be those between infinite parallel plates, which discontinuously drop to zero at the edge of the capacitor. This exercise will give you experience with the use of complex fields in solving physical problems.

a) In the approximation described above we may make the ansatz that the (complex) fields within the capacitor have the form

$$\boldsymbol{E} = \hat{z}f(\rho)e^{-i\omega t}, \qquad \boldsymbol{B} = \boldsymbol{\rho}\times\hat{z}g(\rho)e^{-i\omega t}$$

From the (complex) Maxwell equations determine g in terms of f and show that $f(\rho)$ satisfies the $n = 0$ Bessel equation, whose solution is $f(\rho) = AJ_0(\omega\rho\sqrt{\epsilon\mu})$.

Solution: To write the Maxwell equations we evaluate the curls of the above forms (dropping the $e^{-i\omega t}$ factors:

$$\begin{aligned}
\nabla\times\boldsymbol{E} &= \hat{\rho}\times\hat{z}f'(\rho) = -\hat{\varphi}f'(\rho) \\
\nabla\times\boldsymbol{B} &= \nabla_z(\boldsymbol{\rho}g) - \hat{z}\nabla\cdot(\boldsymbol{\rho}g(\rho)) = -\hat{z}(2g(\rho)+\rho g'(\rho))
\end{aligned}$$

Then the sourceless Maxwell equations read

$$f'(\rho) \;=\; i\omega\rho g(\rho), \qquad -(2g(\rho)+\rho g'(\rho)) = -i\omega\epsilon\mu f \tag{11}$$

The first equation determines $g = f'/i\omega\rho$, which substituted into the second equation gives

$$\begin{aligned} -\frac{2f'}{\rho} - \frac{\rho f'' - f'}{\rho} &= \omega^2\epsilon\mu f \\ f'' + \frac{f'}{\rho} + \omega^2\epsilon\mu f &= 0 \end{aligned} \tag{12}$$

Changing variables to $x = \omega\sqrt{\epsilon\mu}\rho$, we recognize the last equation as the Bessel equation of order $\nu = 0$, with general solution a linear combination of $J_0(\omega\sqrt{\epsilon\mu}\rho)$ and $N_0(\omega\sqrt{\epsilon\mu}\rho)$. However, since there are no sources between the plates, N_0, which is singular as $\rho \to 0$, is not allowed. Thus we concude that

$$f \;=\; AJ_0(\omega\sqrt{\epsilon\mu}\rho), \qquad g = \frac{A\sqrt{\epsilon\mu}}{i\rho}J_0' = -\frac{A\sqrt{\epsilon\mu}}{i\rho}J_1(\omega\sqrt{\epsilon\mu}\rho) \tag{13}$$

b) Now consider the low frequency (quasi-static) limit. Expand the electric and magnetic fields within the capacitor up to quadratic order in ω. Determine A up to this order by demanding charge conservation, $I_0 = -i\omega Q$, where Q is the (complex) charge on the top plate obtained by identifying the surface charge density $\sigma = \epsilon\boldsymbol{n}\cdot\boldsymbol{E}$ in terms of the electric field between the plates. (Here we are neglecting the field outside the plates.)

Solution: From the properties of Bessel fundtions $J_0(x) = 1 - x^2/4 + O(x^4)$ and $J_1(x) = (x/2)(1 - x^2/8 + O(x^4))$. Then

$$\begin{aligned} \boldsymbol{E} &= A\hat{z}\left(1 - \frac{\rho^2\omega^2\epsilon\mu}{4} + O(\omega^4)\right) \\ \boldsymbol{B} &= \frac{iA\omega\epsilon\mu}{2}(\boldsymbol{\rho}\times\hat{z})\left(1 - \frac{\rho^2\omega^2\epsilon\mu}{8} + O(\omega^4)\right) \\ Q &= 2\pi\epsilon A\int_0^a \rho d\rho\left(1 - \frac{\rho^2\omega^2\epsilon\mu}{4} + O(\omega^4)\right) = 2\pi\epsilon A\left(\frac{a^2}{2} - \frac{a^4\omega^2\epsilon\mu}{16} + O(\omega^4)\right) \\ A &= \frac{Q}{\pi a^2\epsilon}\left(1 + \frac{a^2\omega^2\epsilon\mu}{8} + O(\omega^4)\right) = \frac{-I_0}{i\pi a^2\omega\epsilon}\left(1 + \frac{a^2\omega^2\epsilon\mu}{8} + O(\omega^4)\right) \end{aligned} \tag{14}$$

c) Calculate the time averaged electric and magnetic energies by integrating their time averaged densities $u_e = \text{Re}\ \epsilon\boldsymbol{E}\cdot\boldsymbol{E}^*/4$, $u_m = \text{Re}\ \boldsymbol{B}\cdot\boldsymbol{B}^*/4\mu$ over the volume between the capacitor plates.

Solution:

$$\begin{aligned}
U_e &= \frac{\epsilon d}{4}|A|^2 2\pi \int_0^a \rho d\rho \left(1 - \frac{\rho^2\omega^2\epsilon\mu}{2} + O(\omega^4)\right) = \frac{\epsilon d}{4}|A|^2 \pi a^2 \left(1 - \frac{a^2\omega^2\epsilon\mu}{4} + O(\omega^4)\right) \\
&= \frac{d}{4}\frac{|Q|^2}{\pi a^2 \epsilon}(1 + O(\omega^4)) = \frac{\langle Q^2\rangle_{\text{time}}}{2}\frac{d}{\epsilon\pi a^2}(1 + O(\omega^4)) \\
U_m &= \frac{d\omega^2\epsilon^2\mu^2|A|^2\pi}{8\mu}\int_0^a \rho d\rho \left(\rho^2 - \frac{\rho^4\omega^2\epsilon\mu}{4} + O(\omega^4)\right) = \frac{d\omega^2\epsilon^2\mu|A|^2\pi a^4}{32}\left(1 - \frac{a^2\omega^2\epsilon\mu}{6} + O(\omega^4)\right) \\
&= \frac{\mu|I_0|^2 d}{32\pi}\left(1 + \frac{a^2\omega^2\epsilon\mu}{12} + O(\omega^4)\right) = \frac{\mu\langle I^2\rangle_{\text{time}} d}{16\pi}\left(1 + \frac{a^2\omega^2\epsilon\mu}{12} + O(\omega^4)\right)
\end{aligned}$$

d) From the results of part c) read off the capacitance and inductance of this system in the limit $\omega \to 0$. The resonant frequency of a series L, C alternating current circuit is $1/\sqrt{LC}$. Compare the resonant frequency, from your approximate solution to this problem, to a zero of $J_0(\omega a\sqrt{\epsilon\mu})$.

Solution: From the definitions $U_e = Q^2/2C$ and $U_m = LI^2/2$, we read off

$$\begin{aligned}
C &= \frac{\epsilon\pi a^2}{d}(1 + O(\omega^4)), \qquad L = \frac{\mu d}{8\pi}(1 + O(\omega^2)) \\
\omega_{\text{res}} &= \frac{1}{\sqrt{LC}} = \sqrt{\frac{8}{\epsilon\mu a^2}} = 2\sqrt{2}\frac{c}{a} \approx 2.83\frac{c}{a} \qquad (15)
\end{aligned}$$

The first zero of $J_0(\omega a\sqrt{\epsilon\mu})$ is $\omega a\sqrt{\epsilon\mu} = 2.405...$ or $\omega \approx 2.405c/a$, about 15% below our estimate.

44. J, Problem 6.20

Solution:

a)We have

$$\phi = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\boldsymbol{r}', t)}{|\boldsymbol{r} - \boldsymbol{r}'|} = \frac{\delta(t)}{4\pi\epsilon_0}\int dz' \frac{\delta'(z')}{\sqrt{x^2 + y^2 + (z - z')^2}} = \frac{\delta(t)}{4\pi\epsilon_0}\frac{\partial}{\partial z}\frac{1}{r} = -\frac{\delta(t) z}{4\pi\epsilon_0 r^3}$$

b)

$$\boldsymbol{J}_t = \boldsymbol{J} - \epsilon_0\nabla\frac{\partial\phi}{\partial t} = \delta'(t)\left[-\hat{z}\delta(\boldsymbol{r}) + \frac{1}{4\pi}\nabla\frac{z}{r^3}\right] = \delta'(t)\left[-\hat{z}\delta(\boldsymbol{r}) - \frac{1}{4\pi}\nabla\nabla_z\frac{1}{r}\right] \qquad (16)$$

Next examine

$$\begin{aligned}
\nabla_i\nabla_j\frac{1}{r} &= \left(\nabla_i\nabla_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\frac{1}{r} + \frac{1}{3}\nabla^2\frac{1}{r} = (\nabla_i\nabla_j - \frac{1}{3}\delta_{ij}\nabla^2)\frac{1}{r} - \frac{4\pi}{3}\delta(\boldsymbol{r}) \\
&= -\nabla_i\frac{r^i}{r^3} - \frac{4\pi}{3}\delta(\boldsymbol{r}) = \frac{3r^i r^j - r^2\delta_{ij}}{r^5} - \frac{4\pi}{3}\delta(\boldsymbol{r}) \\
\frac{1}{4\pi}\nabla\nabla_z\frac{1}{r} &= \frac{3\boldsymbol{r}z - r^2\hat{z}}{4\pi r^5} - \frac{1}{3}\delta(\boldsymbol{r}) \qquad (17)
\end{aligned}$$

Which then gives

$$\boldsymbol{J}_t = -\delta'(t)\left[\frac{2}{3}\hat{z}\delta(\boldsymbol{r}) + \frac{3\boldsymbol{r}z - r^2\hat{z}}{4\pi r^5}\right] \tag{18}$$

c)We first work out the vector potential, defining $R = |\boldsymbol{r} - \boldsymbol{r}'|$,

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{r}, t) &= \frac{\mu_0}{4\pi}\int d^3x' \frac{\boldsymbol{J}_t(\boldsymbol{r}', t - R/c)}{R} \\
&= -\frac{\mu_0}{4\pi}\int \frac{d^3x'}{R}\delta'(t - R/c)\left[\hat{z}\delta(\boldsymbol{r}') + \frac{1}{4\pi}\nabla'\nabla'_z\frac{1}{r'}\right] \\
&= -\frac{\mu_0}{4\pi}\left[\hat{z}\frac{\delta'(t - r/c)}{r} + \frac{1}{4\pi}\nabla\nabla_z\frac{\partial}{\partial t}\int \frac{d^3x'}{Rr'}\delta(t - R/c)\right]
\end{aligned} \tag{19}$$

where in the last line we integrated the spatial derivatives by parts and used the fact that R depends on $\boldsymbol{r} - \boldsymbol{r}' \equiv \boldsymbol{R}$ to replace the primed derivatives by unprimed ones which were then taken outside the integral. That integral can be best handled by changing variables to $\boldsymbol{R}$:

$$\begin{aligned}
\int \frac{d^3x'}{Rr'}\delta(t - R/c) &= \int \frac{d^3R\delta(t - R/c)}{R|\boldsymbol{r} - \boldsymbol{R}|} = \int \frac{Rd\Omega\delta(t - R/c)}{|\boldsymbol{r} - \boldsymbol{R}|} = 4\pi c^2 t\theta(t)\frac{1}{r_>} \\
&= 4\pi c\theta(t)\left[\theta(ct - r) + \theta(r - ct)\frac{ct}{r}\right]
\end{aligned} \tag{20}$$

where $r_>$ is the larger of r, ct. The angular integral follows by expanding $|\boldsymbol{r} - \boldsymbol{R}|^{-1}$ in Legendre polynomials. The time derivative is

$$\begin{aligned}
\frac{\partial}{\partial t}\int \frac{d^3x'}{Rr'}\delta(t - R/c) &= \frac{4\pi c^2}{r}\theta(t)\theta(r - ct) \\
\boldsymbol{A} &= -\frac{\mu_0}{4\pi}\left[\hat{z}\frac{\delta'(t - r/c)}{r} + c^2\theta(t)\nabla\nabla_z\frac{\theta(r - ct)}{r}\right]
\end{aligned} \tag{21}$$

The acausal second term doesn't contribute to the magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ because it is a gradient. It does contribute to the electric field:

$$\begin{aligned}
\boldsymbol{E} &= -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \\
&= \frac{\delta(t)}{4\pi\epsilon_0}\nabla\frac{z}{r^3} + \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\left[\hat{z}\frac{\delta'(t - r/c)}{r} + c^2\theta(t)\nabla\nabla_z\frac{\theta(r - ct)}{r}\right] \\
&= \frac{\delta(t)}{4\pi\epsilon_0}\nabla\frac{z}{r^3} + \frac{\mu_0}{4\pi}\left[\hat{z}\frac{\delta''(t - r/c)}{r} + c^2\delta(t)\nabla\nabla_z\frac{1}{r} - c^2\nabla\nabla_z\frac{\delta(t - r/c)}{r}\right] \\
&= \frac{\mu_0}{4\pi}\left[\hat{z}\frac{\delta''(t - r/c)}{r} - c^2\nabla\nabla_z\frac{\delta(t - r/c)}{r}\right] = \frac{c}{4\pi\epsilon_0}\left[\hat{z}\frac{\delta''(r - ct)}{r} - \nabla\nabla_z\frac{\delta(r - ct)}{r}\right]
\end{aligned}$$

Notice that the acausal terms (those proportional to $\delta(t)$) from ϕ and $\boldsymbol{A}$ have cancelled! Writing out the components of $\boldsymbol{E}$:

$$\begin{aligned}
E_x &= -\frac{c}{4\pi\epsilon_0}\nabla_x\nabla_z\frac{\delta(r-ct)}{r} = -\frac{c}{4\pi\epsilon_0}\nabla_x\left[z\frac{\delta'(r-ct)}{r^2} - z\frac{\delta(r-ct)}{r^3}\right] \\
&= -\frac{c}{4\pi\epsilon_0}\frac{xz}{r}\left[\frac{\delta''(r-ct)}{r^2} - 3\frac{\delta'(r-ct)}{r^3} + 3\frac{\delta(r-ct)}{r^4}\right] \\
&= -\frac{c}{4\pi\epsilon_0}\sin\theta\cos\varphi\cos\theta\left[\frac{\delta''(r-ct)}{r} - 3\frac{\delta'(r-ct)}{r^2} + 3\frac{\delta(r-ct)}{r^3}\right] \\
E_y &= -\frac{c}{4\pi\epsilon_0}\sin\theta\sin\varphi\cos\theta\left[\frac{\delta''(r-ct)}{r} - 3\frac{\delta'(r-ct)}{r^2} + 3\frac{\delta(r-ct)}{r^3}\right] \\
E_z &= \frac{c}{4\pi\epsilon_0}\left[\frac{\delta''(r-ct)}{r} - \nabla_z\nabla_z\frac{\delta(r-ct)}{r}\right] \\
&= \frac{c}{4\pi\epsilon_0}\left[\frac{\delta''(r-ct)}{r} - \nabla_z\left(z\frac{\delta'(r-ct)}{r^2} - z\frac{\delta(r-ct)}{r^3}\right)\right] \\
&= \frac{c}{4\pi\epsilon_0}\left[\frac{\delta''(r-ct)}{r} - \left(\frac{\delta'(r-ct)}{r^2} - \frac{\delta(r-ct)}{r^3}\right)\right] \\
&\quad -\frac{c}{4\pi\epsilon_0}\frac{z^2}{r}\left[\frac{\delta''(r-ct)}{r^2} - 3\frac{\delta'(r-ct)}{r^3} + 3\frac{\delta(r-ct)}{r^4}\right] \\
&= \frac{c}{4\pi\epsilon_0}\left[\sin^2\theta\frac{\delta''(r-ct)}{r} + (3\cos^2\theta - 1)\left(\frac{\delta'(r-ct)}{r^2} - \frac{\delta(r-ct)}{r^3}\right)\right]
\end{aligned}$$

Both $\boldsymbol{B}$ and $\boldsymbol{E}$ have contirbutions only for $r = ct$ consistent with causality.