Electromagnetic Theory I

Solution Set 12

Due: 9 December 2020

45. Some derivations in the lecture notes.

a) Equation (336) of the lecture notes gives a sequence of steps that leads to the identification of the momentum carried by the fields. Explain each step and discuss our identification of $\boldsymbol{g}$ with the momentum density of the fields.

Solution:

1. First equality uses Maxwell equations to eliminate ρ, $\boldsymbol{J}$ in favor of the fields.
2. Second equality rewrites the last term as a time derivative on the product minus a term containing the time derivative of $\boldsymbol{B}$ which is replaced by the curl of $\boldsymbol{E}$. At the same time the triple vector product of the second term is expanded,
3. In the third equality the triple vector product in the previous second term is expanded.
4. The fourth equality recognizes the second and fourth terms of the previous step as a total derivative and then rewrites the fifth term as a total derivative minus a term involving the divergence of $\boldsymbol{B}$ which the fifth equality recognizes is zero by one of the Maxwell equations.
5. The last line is explained in the following text.

b) Show that G given in Eq (348) solves Eq.(347). Note: these equation numbers are from a recent update of the lecture notes. In the previous version they were (346) and (345).

Solution: Set $\boldsymbol{r}' = 0$ Then

$$4\pi\nabla G_{\pm} = e^{\pm i\omega r/c}\nabla\frac{1}{r} \pm i\omega\frac{\boldsymbol{r}}{cr^2}e^{\pm i\omega r/c} \quad (1)$$

$$4\pi\nabla^2 G_{\pm} = e^{\pm i\omega r/c}\nabla^2\frac{1}{r} \pm i\omega\frac{\boldsymbol{r}}{cr}e^{\pm i\omega r/c}\cdot\nabla\frac{1}{r} \pm i\omega\nabla\cdot\left(\frac{\boldsymbol{r}}{cr^2}\right)e^{\pm i\omega r/c} - \frac{\omega^2}{c^2}\frac{e^{i\omega r/c}}{r} \quad (2)$$

$$= -4\pi\delta(\boldsymbol{r}) - \frac{\omega^2}{c^2}4\pi G_{\pm} \quad (3)$$

which is a rearrangement of the desired result. In the last step we used

$$\nabla\cdot\left(\frac{\boldsymbol{r}}{cr^2}\right) = \frac{3}{cr^2} - 2\frac{\boldsymbol{r}\cdot\boldsymbol{r}}{cr^4} = \frac{1}{cr^2} \quad (4)$$

to show the cancellation of two terms.

46. Plane wave solutions of Maxwell's Equations

a) Plugging the (complex) plane wave ansatz

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t}, \qquad \boldsymbol{B} = \boldsymbol{B}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t} \tag{5}$$

into the sourceless vacuum Maxwell equations determine $\boldsymbol{B}_0$ and ω in terms of $\boldsymbol{E}_0$ and $\boldsymbol{k}$, and spell out any further constraints on $\boldsymbol{E}_0$. Real solutions can be constructed from these complex ones by taking their real parts. Why do we know that the real parts will be solutions?

Solution: In Maxwell's equations the plane wave space-time dependence means $\nabla \to i\boldsymbol{k}$, and $\partial/\partial t \to -i\omega$. Then we have

$$\boldsymbol{k}\cdot\boldsymbol{E}_0 = \boldsymbol{k}\cdot\boldsymbol{B}_0 = 0, \qquad \boldsymbol{B}_0 = \frac{\boldsymbol{k}\times\boldsymbol{E}_0}{\omega}, \qquad \epsilon_0\mu_0\boldsymbol{E}_0 = -\frac{\boldsymbol{k}\times\boldsymbol{B}_0}{\omega} \tag{6}$$

Plugging the third equation into the fourth one gives

$$\boldsymbol{E}_0 \;=\; -c^2\frac{\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}_0)}{\omega^2} = \frac{c^2k^2}{\omega^2}\boldsymbol{E}_0, \qquad \omega = |\boldsymbol{k}|c, \qquad \boldsymbol{B}_0 = \frac{\boldsymbol{k}\times\boldsymbol{E}_0}{kc} \tag{7}$$

$\boldsymbol{E}_0$ can be any complex vector perpendicular to $\boldsymbol{k}$: $\boldsymbol{k}\cdot\boldsymbol{E}_0 = 0$. Since Maxwell's equations are real $\boldsymbol{E}^*, \boldsymbol{B}^*$ are solutions if $\boldsymbol{E}, \boldsymbol{B}$ are. The real parts are just linear superpositions of these.

b) Evaluate the energy and momentum densities of the real plane wave solutions (including the direction of the momentum which is a vector!). Also give the time average of these densities.

Solution:

$$u \;=\; \frac{\epsilon_0}{8}(\boldsymbol{E}+\boldsymbol{E}^*)^2 + \frac{1}{8\mu_0}(\boldsymbol{B}+\boldsymbol{B}^*)^2 = \frac{\epsilon_0}{8}\left[(\boldsymbol{E}+\boldsymbol{E}^*)^2 + c^2(\boldsymbol{B}+\boldsymbol{B}^*)^2\right] \tag{8}$$

It follows from $c\boldsymbol{B} = \hat{k}\times\boldsymbol{E}$ that $c^2\boldsymbol{B}^2 = \boldsymbol{E}^2$ and $c^2\boldsymbol{B}\cdot\boldsymbol{B}^* = \boldsymbol{E}\cdot\boldsymbol{E}^*$ so the magnetic term is equal to the electric term and

$$\begin{aligned} u \;&=\; \frac{\epsilon_0}{4}(\boldsymbol{E}+\boldsymbol{E}^*)^2 = \frac{\epsilon_0}{2}(\boldsymbol{E}\cdot\boldsymbol{E}^* + \mathrm{Re}\boldsymbol{E}^2) = \frac{\epsilon_0}{2}(\boldsymbol{E}_0\cdot\boldsymbol{E}_0^* + \mathrm{Re}\boldsymbol{E}_0^2 e^{2i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}) \\ \langle u\rangle_t \;&=\; \frac{\epsilon_0}{2}\boldsymbol{E}_0\cdot\boldsymbol{E}_0^* \end{aligned} \tag{9}$$

Similarly

$$\begin{aligned} \boldsymbol{g} \;&=\; \epsilon_0\mathrm{Re}\boldsymbol{E}\times\mathrm{Re}\boldsymbol{B} = \frac{\epsilon_0}{c}\mathrm{Re}\boldsymbol{E}\times(\hat{k}\times\mathrm{Re}\boldsymbol{E}) = \frac{\epsilon_0}{c}\hat{k}(\mathrm{Re}\boldsymbol{E})^2 = \frac{u}{c}\hat{k} \\ \langle\boldsymbol{g}\rangle_t \;&=\; \frac{\epsilon_0}{2c}\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*\hat{k} \end{aligned} \tag{10}$$

c) J, Problem 6.11 a)

Solution: The momentum per unit area per unit time absorbed by the screen is $c\boldsymbol{g}$. From $\boldsymbol{F} = d\boldsymbol{p}/dt$ it follows that $c\boldsymbol{g}$ is the force per unit area exerted on the screen. Then Pressure=Force/Area= $c|\boldsymbol{g}| = u = \epsilon_0|\boldsymbol{E}_0|^2/2$ on time average.

d) J, Problem 6.11 b)

Solution: The pressure is

$$u = \frac{S}{c} = \frac{1.4\text{kW/m}^2}{3\times 10^8\text{m/s}} = 4.7\times 10^{-6}\text{J/m}^3 = 4.7\times 10^{-6}\text{N/m}^2 \tag{11}$$

The mass per unit area is 10^{-3}kg/m^2 so the acceleration is $a = 4.7\times 10^{-3}\text{m/s}^2 = 4.7\text{mm/s}^2$.

47. J, Problem 7.2

Solution:

a) Take the n_1n_2 interface to be the xy-plane and the n_2n_3 interface to be the parallel plane at $z = d$. Take $\boldsymbol{E}$ in the x direction and $\boldsymbol{B}$ in the y direction. Parameterize the fields in the three regions as

$$\begin{aligned}
E_I &= [E_ie^{ik_1z} + E_re^{-ik_1z}]e^{-i\omega t}, \qquad k_1 = n_1\frac{\omega}{c} &(12)\\
E_{II} &= [E_1e^{ik_2z} + E_2e^{-ik_2z}]e^{-i\omega t}, \qquad k_2 = n_2\frac{\omega}{c} &(13)\\
E_{III} &= E_te^{ik_3z}e^{-i\omega t}, \qquad k_3 = n_3\frac{\omega}{c} &(14)
\end{aligned}$$

The magnetic field is determined by $i\omega B = \nabla_z E$:

$$\begin{aligned}
B_I &= \frac{k_1}{\omega}[E_ie^{ik_1z} - E_re^{-ik_1z}]e^{-i\omega t}, \qquad k_1 = n_1\frac{\omega}{c} &(15)\\
B_{II} &= \frac{k_2}{\omega}[E_1e^{ik_2z} - E_2e^{-ik_2z}]e^{-i\omega t}, \qquad k_2 = n_2\frac{\omega}{c} &(16)\\
B_{III} &= \frac{k_3}{\omega}E_te^{ik_3z}e^{-i\omega t}, \qquad k_3 = n_3\frac{\omega}{c} &(17)
\end{aligned}$$

Next we impose continuity of H, E at the interfaces.

$$\begin{aligned}
E_i + E_r &= E_1 + E_2, \qquad E_1e^{ik_2d} + E_2e^{-ik_2d} = E_te^{ik_3d}\\
n_1(E_i - E_r) &= n_2(E_1 - E_2), \qquad n_2(E_1e^{ik_2d} - E_2e^{-ik_2d}) = n_3E_te^{ik_3d} &(18)
\end{aligned}$$

The n_2n_3 equations determine

$$E_1 = \frac{n_2+n_3}{2n_2}E_te^{i(k_3-k_2)d}, \qquad E_2 = \frac{n_2-n_3}{2n_2}E_te^{i(k_3+k_2)d} \tag{19}$$

Plugging these into the $n_1 n_2$ equations gives

$$\begin{aligned}
E_i + E_r &= E_t e^{ik_3 d}(\cos k_2 d - i\frac{n_3}{n_2}\sin k_2 d) \\
E_i - E_r &= \frac{n_2}{n_1} E_t e^{ik_3 d}(-i\sin k_2 d + \frac{n_3}{n_2}\cos k_2 d) \\
E_t &= \frac{2e^{-ik_3 d}}{(1+n_3/n_1)\cos k_2 d - i(n_3/n_2 + n_2/n_1)\sin k_2 d} E_i \\
E_r &= \frac{(1-n_3/n_1)\cos k_2 d - i(n_3/n_2 - n_2/n_1)\sin k_2 d}{(1+n_3/n_1)\cos k_2 d - i(n_3/n_2 + n_2/n_1)\sin k_2 d} E_i
\end{aligned} \tag{20}$$

The time averaged Poynting vectors are

$$\boldsymbol{S}_i = \hat{z}\frac{k_1}{2\omega\mu_0}|E_i|^2, \qquad \boldsymbol{S}_r = -\hat{z}\frac{k_1}{2\omega\mu_0}|E_r|^2, \qquad \boldsymbol{S}_t = \hat{z}\frac{k_3}{2\omega\mu_0}|E_t|^2 \tag{21}$$

from which the reflection and transmission intensities are

$$\begin{aligned}
R &= \frac{(1-n_3/n_1)^2\cos^2 k_2 d + (n_3/n_2 - n_2/n_1)^2 \sin^2 k_2 d}{(1+n_3/n_1)^2\cos^2 k_2 d + (n_3/n_2 + n_2/n_1)^2 \sin^2 k_2 d} \\
T &= \frac{4(n_3/n_1)}{(1+n_3/n_1)^2\cos^2 k_2 d + (n_3/n_2 + n_2/n_1)^2 \sin^2 k_2 d}
\end{aligned} \tag{22}$$

it is straightforward to check that $R+T=1$ as conservation of energy requires. Putting in the numbers for the special cases mentioned gives

$$\begin{aligned}
T &= \frac{48}{64 - 15\sin^2 k_2 d}, \qquad R = \frac{16 - 15\sin^2 k_2 d}{64 - 15\sin^2 k_2 d}, \qquad (n_1, n_2, n_3) = (1, 2, 3) \\
T &= \frac{48}{64 - 15\sin^2 k_2 d}, \qquad R = \frac{16 - 15\sin^2 k_2 d}{64 - 15\sin^2 k_2 d}, \qquad (n_1, n_2, n_3) = (3, 2, 1) \\
T &= \frac{32}{36 + 45\sin^2 k_2 d}, \qquad R = \frac{4 + 45\sin^2 k_2 d}{36 + 45\sin^2 k_2 d}, \qquad (n_1, n_2, n_3) = (2, 4, 1)
\end{aligned}$$

The equality of the first 2 cases is due to time reversal invariance.

b) To have $R = 0$ requires both terms in the numerator to vanish.

$$(1 - n_3/n_1)^2 \cos^2 k_2 d = (n_3/n_2 - n_2/n_1)^2 \sin^2 k_2 d = 0 \tag{23}$$

If neither the sine nor the cosine are zero this would require $n_1 = n_2 = n_3$, which means the absence of an interface at all. If $n_1 = n_3 \neq n_2$ it happens if $k_2 d = n\pi$. Finally, if $n_1 \neq n_3$ which is the case of interest $R = 0$ requires both $n_2 = \sqrt{n_1 n_3}$ and $k_2 d = \pi/2 + n\pi$, which means $\omega = \pi c(2n+1)/2n_2 d$.

48. J, Problem 7.3, but only for the case of polarization perpendicular to the plane of incidence.

Solution:

a) Put the boundaries of the gap at $z = 0$ and $z = d$, and take the incident beam coming from the region $z < 0$. Take the yz-plane as the plane of incidence, so that $\boldsymbol{E} = E\hat{x}$. Then we put

$$E = e^{ik_\perp y - i\omega t}\begin{cases} E_i e^{ikz} + E_r e^{-ikz} & z < 0 \\ E_1 e^{ik'z} + E_2 e^{-ik'z} & 0 < z < d \\ E_t e^{ikz} & d < z \end{cases} \tag{24}$$

Here $k^2 = n^2\omega^2/c^2 - k_\perp^2$ and $k'^2 = \omega^2/c^2 - k_\perp^2$. Continuity of E at the interfaces implies

$$E_i + E_r = E_1 + E_2, \qquad E_1 e^{ik'd} + E_2 e^{-ik'd} = E_t e^{ikd} \tag{25}$$

The magnetic field is given by $\boldsymbol{B} = \nabla \times \boldsymbol{E}/i\omega = \boldsymbol{k} \times \boldsymbol{E}/\omega$. In the slabs $\boldsymbol{k} = k_\perp \hat{y} \pm k\hat{z}$ and $\boldsymbol{k} \times \hat{x} = -k_\perp \hat{z} \pm k\hat{y}$, whereas in the gap, $\boldsymbol{k} \times \hat{x} = -k_\perp \hat{z} \pm k'\hat{y}$. Thus continuity of the normal $\boldsymbol{B}$ field which is B_z is automatic given the continuity of E. On the other hand continuity of tangential $\boldsymbol{H} = \boldsymbol{B}/\mu_0$ gives new information:

$$k(E_i - E_r) = k'(E_1 - E_2), \qquad k'E_1 e^{ik'd} - k'E_2 e^{-ik'd} = kE_t e^{ikd} \tag{26}$$

These 4 equations determine E_1, E_2, E_t, E_r in terms of E_i:

$$\begin{aligned} E_i &= \frac{k+k'}{2k}E_1 + \frac{k-k'}{2k}E_2, \qquad E_r = \frac{k-k'}{2k}E_1 + \frac{k+k'}{2k}E_2 \\ E_1 &= \frac{k'+k}{2k'}E_t e^{i(k-k')d}, \qquad E_2 = \frac{k'-k}{2k'}E_t e^{i(k+k')d} \\ E_t &= E_i e^{-ikz}\frac{4kk'}{(k+k')^2 e^{-ik'd} - (k-k')^2 e^{ik'd}} \\ E_r &= E_i \frac{(k^2-k'^2)e^{-ik'd} - (k^2-k'^2)e^{ik'd}}{(k+k')^2 e^{-ik'd} - (k-k')^2 e^{ik'd}} \end{aligned} \tag{27}$$

The ratios of the intensities are then

$$\begin{aligned} \frac{|E_t|^2}{|E_i|^2} &= \left|\frac{4kk'}{(k+k')^2 e^{-ik'd} - (k-k')^2 e^{ik'd}}\right|^2 \\ \frac{|E_r|^2}{|E_i|^2} &= \left|\frac{(k^2-k'^2)e^{-ik'd} - (k^2-k'^2)e^{ik'd}}{(k+k')^2 e^{-ik'd} - (k-k')^2 e^{ik'd}}\right|^2 \end{aligned}$$

Note that $k^2 = n^2\omega^2(1 - \sin^2\theta)$ is positive for all angles of incidence θ so k is generally real. But k'^2 is negative (implying imaginary k') for $n^2\sin^2\theta > 1$, i.e. for total reflection at th first interface. In either case, however, it is straightforward to check that $|E_r|^2 + |E_t|^2 = |E_i|^2$.

b) For an incident angle greater than the critical angle, $k' \equiv i\kappa = i(\omega/c)\sqrt{n^2 \sin^2\theta_i - 1} = (2\pi i/\lambda)\sqrt{n^2\sin^2\theta_i - 1}$ is imaginary. Then

$$\begin{aligned}
\frac{|E_t|^2}{|E_i|^2} &= \frac{16k^2\kappa^2}{|(k+i\kappa)^2 e^{\kappa d} - (k-i\kappa)^2 e^{-\kappa d}|^2} = \frac{16k^2\kappa^2}{|2(k^2-\kappa^2)\sinh\kappa d + 4ik\kappa\cosh\kappa d|^2} \\
&= \frac{16k^2\kappa^2}{4(k^2-\kappa^2)^2\sinh^2\kappa d + 16k^2\kappa^2\cosh^2\kappa d} = \frac{1}{1+[(k^2+\kappa^2)^2/4\kappa^2k^2]\sinh^2\kappa d} \\
&\sim \frac{16\kappa^2k^2}{(k^2+\kappa^2)^2}e^{-2\kappa d}
\end{aligned}$$

for $d/\lambda \gg 1$. Of course the ratio goes to 1 when $d \to 0$. We have

$$\begin{aligned}
\frac{(k^2+\kappa^2)^2}{4\kappa^2k^2} &= \frac{(n^2-1)^2}{4n^2\cos^2\theta_i(n^2\sin^2\theta_i - 1)} \\
\kappa d &= \frac{2\pi d}{\lambda}\sqrt{n^2\sin^2\theta_i - 1} \qquad (28)
\end{aligned}$$

This behavior is analogous to quantum mechanical tunneling through a barrier. For purposes of the plot, pick $\theta_i = \pi/4$ and $n = 2$, so

$$T \equiv \frac{|E_t|^2}{|E_i|^2} = \frac{1}{1+[9/8]\sinh^2 2\pi d/\lambda} \qquad (29)$$