Electromagnetic Theory II

Solution Set 1

Due: 20 January 2021

1. J, Problem 11.4.

Solution:

a) The upward light pulse of the moving system travels from the point $(0,0)$ to the point (vt, d) a total distance $\sqrt{d^2+v^2t^2}$ which takes time $t = \sqrt{d^2+v^2t^2}/c$ or $t^2(c^2-v^2) = d^2$ so $t = d/\sqrt{c^2-v^2} = \gamma d/c$. The return pulse requires the same amount of time so the stationary observer sees the time of tick to be $2t = (2d/c)/\sqrt{1-v^2/c^2}$ a factor of γ larger than the tick of the clock at rest.

b) Because of length contraction, the system moving parallel to the line from PF to M appears to have a length d/γ. The upward pulse then has to travel a distance $d/\gamma + vt$ taking a time $t_1 = (d/\gamma + vt_1)/c$ or $t_1 = d/(c-v)\gamma$. The downward jouney is a distance $d/\gamma - vt_2$ or time $t_2 = d/(c+v)\gamma$. The total tick is therefore $t_1 + t_2 = 2dc/(c^2-v^2)\gamma = 2\gamma d/c$, the same as in part a).

2. J, Problem 11.7. In this problem the starting shots can be thought of as two events in spacetime. You are asked to find a Lorentz frame for which, depending on the relationship of T to d/c, these events either occur at the same time at different points in space or at the same point in space at different times. *Hint*: consider Lorentz boosts in the y direction, to find these frames. Add a third part

Solution;

a) Consider the two shots as event 1 with spacetime coordinates $(0,0,0,0)$ and event 2 with spacetime coordinates $(0,d,0,T)$. The invariant interval between these events is $s^2 = d^2 - c^2T^2$. If $T < d/c$ this interval is space-like, which means there is a frame in which the two events are simultanteous. In this frame there is no handicap, so in fact there is no true handicap. On the other hand if $T > d/c$, the interval is time-like, which means there is a frame in which the events occur at the same space point, but at different times. In this frame both sprinters depart from the same point at different times and there is a true handicap.

b) We now find the K' coordinates for the respective frames defined in a). We write out a Lorentz boost in the y direction:

$$x' = x, \qquad y' = \gamma(y - vt), \qquad z' = z, \qquad t' = \gamma(t - vy/c^2) \tag{1}$$

The primed coordinates of event 1 are $(0,0,0,0)$ and of event 2 are $(0, \gamma(d-vT), 0, \gamma(T - vd/c^2)$. In this frame the events will be simultaneous if $v = c^2T/d$ which will be less than c provided that $T < d/c$, the condition for a space-like interval. The events will occur at the same space point if $v = d/T$ which is less than c if $T > d/c$, the condition for a time-like interval. The starting primed spacetime coordinates of the two sprinters are $(0, 0, 0, 0)$, $(0, \sqrt{(}d^2 - c^2T^2), 0, 0)$ in the case $T < d/c$; and $(0, 0, 0, 0)$, $(0, 0, 0, \sqrt{T^2 - d^2/c^2})$ in the case $T > d/c$.

c) In frame K the trajectories of the two sprinters can be taken as $(x_1(t), y_1(t)) = (v_1 t, 0, 0)$ and $(x_2(t), y_2(t)) = (v_2(t - T), d, 0)$. Transform these trajectories to the K' frame in both cases. That is work out $(x'_{1,2}(t'), y'_{1,2}(t'))$.

Solution:

$$\begin{aligned}
x'_1 &= x_1 = v_1 t, \qquad t' = \gamma t, \qquad x'_1(t') = \frac{v_1 t'}{\gamma}, \qquad y'_1(t') = -\gamma v t = -v t' && (2)\\
x'_2 &= x_2 = v_2(t - T), \qquad t' = \gamma(t - dv/c^2), \qquad x'_2(t') = v_2(t'/\gamma + dv/c^2 - T)\\
y'_2(t') &= \gamma(d - vt) = \gamma(d - v(t'/\gamma + dv/c^2) = -vt' + d\sqrt{1 - v^2/c^2} && (3)
\end{aligned}$$

For $T < d/c$, $v = c^2T/d$ and $\gamma = 1/\sqrt{1 - c^2T^2/d^2}$, so

$$\begin{aligned}
(x'_1(t'), y'_1(t')) &= (v_1 t'\sqrt{1 - c^2T^2/d^2}, -c^2Tt'/d) && (4)\\
(x'_2(t'), y'_2(t')) &= (v_2 t'\sqrt{1 - c^2T^2/d^2}, -c^2Tt'/d + d\sqrt{1 - c^2T^2/d^2}) && (5)
\end{aligned}$$

We see that both sprinters set off at the same time and the speedier one stays ahead of the other at all times. For $T > d/c$, $v = d/T$ and $\gamma = 1/\sqrt{1 - d^2/c^2T^2}$, so

$$\begin{aligned}
(x'_1(t'), y'_1(t')) &= (v_1\sqrt{1 - d^2/c^2T^2}, -d/T)t' && (6)\\
(x'_2(t'), y'_2(t')) &= (v_2\sqrt{1 - d^2/c^2T^2}, -d/T)(t' - T\sqrt{1 - d^2/c^2T^2}) && (7)
\end{aligned}$$

Here we see that sprinter 1 (assumed slower, $v_1 < v_2$) starts out from the origin at $t' = 0$, and sprinter 2 starts out from the origin at the later time $t' = T\sqrt{1 - d^2/c^2T^2}$. their separation in the y' direction is constant in time, but in the x' direction sprinter 2 catches sprinter 1 at time $t' = T_c$ given by

$$v_2(T_c - T\sqrt{1 - d^2/c^2T^2}) = v_1 T_c, \qquad T_c = \frac{Tv_2\sqrt{1 - d^2/c^2T^2}}{v_2 - v_1} \qquad (8)$$

3. One way to compare length measurements in different frames is to compare how a wave packet looks in different frames. Let the function $f(x)$, independent of y, z, describe a symmetric profile in the x coordinate with a definite width w. Then $\psi(x, t) = f(x - ct)$ solves the wave equation

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0 \qquad (9)$$

in the unprimed frame.

a) Find the function $\psi'(x', t')$ that describes how this packet looks in the primed frame which is moving in the negative x direction at velocity v, and show that it solves the wave equation in the primed system.

Solution: The Lorentz transform is $x = \gamma(x' - vt')$, $t = \gamma(t' - vx'/c^2)$, so $x - ct = \gamma(1 + v/c)(x' - ct')$. Thus $\psi'(x', t') = f(\gamma(1 + v/c)(x' - ct')$. Because it is a function of $x' - ct'$ it automatically solves the wave equation in the primed system.

b) What is the width w' of this profile as determined in the primed system? Notice that w'/w is *not* the usual Lorentz contraction factor.

Solution: All dimensions of the profile are scaled by the factor $\gamma(1+v/c)$. In particular $w = \gamma(1 + v/c)w'$, or $w' = w\sqrt{(c - v)/(c + v)}$.

c) To understand this difference, consider a rod of rest length L_0 parallel to the x-axis, moving in the positive x direction with velocity u in the unprimed frame. What is its velocity in the primed frame?

Solution: The velocity of the rod in the primed system is the relativistic combination of velocities v and u: $u' = (u + v)/(1 + uv/c^2)$.

d) Calculate the lengths L, L' of the rod as seen in the unprimed and primed systems respectively. Compare L'/L to w'/w, and comment.

Solution: According to the length contraction formula $L = L_0\sqrt{1 - u^2/c^2}$, and $L' = L_0\sqrt{1 - u'^2/c^2}$. Then

$$\begin{aligned} \frac{L'}{L} &= \sqrt{\frac{1 - u'^2/c^2}{1 - u^2/c^2}} = \frac{1}{1 + uv/c^2}\sqrt{\frac{1 + 2uv/c^2 + u^2v^2/c^4 - (u^2 + v^2 + 2uv)/c^2}{1 - u^2/c^2}} \\ &= \frac{\sqrt{1 - v^2/c^2}}{1 + uv/c^2} \to \sqrt{\frac{c - v}{c + v}} \end{aligned} \tag{10}$$

where the right side of the last line is the limit $u \to c$. In this later limit $L'/L \to w'/w$. Thus we can understand the ratio of widths of wave packets in two Lorentz frames in terms of the length contraction formulas for a rod moving at the speed of light in both frames.

4. We have represented a general Lorentz transformation as a 4×4 matrix $\Lambda^\mu{}_\nu$. An infinitesimal Lorentz transformation can be written $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon^\mu{}_\nu$ where the $\epsilon^\mu{}_\nu \ll 1$.

a) Show that the restriction on Λ required of Lorentz transformations is equivalent to the statement that the matrix $\epsilon_{\mu\nu} = \eta_{\mu\rho}\epsilon^\rho{}_\nu$ is antisymmetric $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$. In particular, this means $\epsilon^0{}_i = +\epsilon^i{}_0$, $i = 1, 2, 3$ and $\epsilon^i{}_j = -\epsilon^j{}_i$, $i, j = 1, 2, 3$.

Solution: The condition that the matrix ${\Lambda^\mu}_\nu$ describe a Lorentz transformation is $\eta_{\rho\sigma} = \eta_{\mu\nu}{\Lambda^\mu}_\rho{\Lambda^\nu}_\sigma$. Inserting the infinitesimal form into this gives

$$\begin{aligned}
\eta_{\rho\sigma} &= \eta_{\mu\nu}({\delta^\mu}_\rho + {\epsilon^\mu}_\rho)({\delta^\nu}_\sigma + {\epsilon^\nu}_\sigma) = \eta_{\mu\nu}({\delta^\mu}_\rho{\delta^\nu}_\sigma + {\delta^\nu}_\sigma{\epsilon^\mu}_\rho + {\delta^\mu}_\rho{\epsilon^\nu}_\sigma + O(\epsilon^2)) \\
&= \eta_{\rho\sigma} + \eta_{\mu\sigma}{\epsilon^\mu}_\rho + \eta_{\rho\nu}{\epsilon^\nu}_\sigma + O(\epsilon^2) = \eta_{\rho\sigma} + \epsilon_{\rho\sigma} + \epsilon_{\sigma\rho} + O(\epsilon^2) \qquad (11)
\end{aligned}$$

It immediately follows that $\epsilon_{\rho\sigma} + \epsilon_{\sigma\rho} = 0$ or $\epsilon_{\rho\sigma} = -\epsilon_{\sigma\rho}$ as desired.

b) For an infinitesimal boost in the x direction the matrix ϵ^μ_ν is therefore

$$\epsilon \equiv \lambda K_x = \lambda \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (12)$$

By explicitly evaluating the expansion

$$\Lambda = e^{\lambda K_x} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} K_x^n$$

when λ is *not* small, show that the matrix Λ gives a finite boost in the x-direction, and identify the boost velocity.

Solution: We first note that

$$\begin{aligned}
K_x^2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
K_x^{2n} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad n > 0 \\
K_x^{2n+1} &= K_x, \qquad n \geq 0 \qquad (13)
\end{aligned}$$

Now $\sum_{n=0}^{\infty} \lambda^{2n}/(2n)! = \cosh\lambda$ and $\sum_{n=0}^{\infty} \lambda^{2n+1}/(2n+1)! = \sinh\lambda$, so the expansion reads

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} K_x^n = I - K_x^2 + K_x^2 \cosh\lambda + K_x \sinh\lambda = \begin{pmatrix} \cosh\lambda & \sinh\lambda & 0 & 0 \\ \sinh\lambda & \cosh\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which describes a Lorentz transformation parallel to the x axis with boost velocity $v = -c\tanh\lambda$,

c) Similarly show that a finite rotation about the z-axis is the exponential of an infinitesimal rotation.

Solution: For an infinitesimal rotation in the xy plane the matrix $\epsilon^{\mu}{}_{\nu}$ is

$$\epsilon \equiv \theta \mathcal{J}_z = \theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{14}$$

The antisymmetry means that

$$\begin{aligned} \mathcal{J}_z^2 &= -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \mathcal{J}_z^{2n} &= (-)^n \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad n > 0 \\ \mathcal{J}_z^{2n+1} &= (-)^n \mathcal{J}_z \end{aligned} \tag{15}$$

So this time $\sum_{n=0}^{\infty}(-)^n\theta^{2n}/(2n)! = \cos\theta$ and $\sum_{n=0}^{\infty}(-)^n\theta^{2n+1}/(2n+1)! = \sin\theta$, and we find

$$e^{\theta\mathcal{J}_z} = \sum_{n=0}^{\infty}\frac{\theta^n}{n!}\mathcal{J}_z^n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which precisely describes a rotation by angle θ about the z-axis.