Electromagnetic Theory II

Solution Set 2

Due: 27 January 2021

5. Starting with the formula for combining the velocities $v\hat{x}, \boldsymbol{V}$ relativistically

$$U_x = \frac{v+V_x}{1+vV_x/c^2}, \qquad U_y = \frac{V_y}{\gamma(1+vV_x/c^2)}, \qquad U_z = \frac{V_z}{\gamma(1+vV_x/c^2)} \tag{1}$$

Show that $\boldsymbol{U}^2$ is always less than c^2, as long as the individual velocities $v\hat{x}, \boldsymbol{V}$ are less than c. Discuss the limits when one or both approach that of light.

Solution: We simply evaluate

$$\begin{aligned}
\boldsymbol{U}^2 &= \frac{(v+V_x)^2+(V_y^2+V_z^2)(1-v^2/c^2)}{(1+vV_x/c^2)^2} = \frac{(v+V_x)^2+(\boldsymbol{V}^2-V_x^2)(1-v^2/c^2)}{(1+vV_x/c^2)^2} \\
&= \frac{v^2+2vV_x+v^2V_x^2/c^2+\boldsymbol{V}^2(1-v^2/c^2)}{(1+vV_x/c^2)^2} = \frac{v^2-c^2+c^2(1+vV_x/c^2)^2+\boldsymbol{V}^2(1-v^2/c^2)}{(1+vV_x/c^2)^2} \\
&= c^2 - c^2\frac{(1-\boldsymbol{V}^2/c^2)(1-v^2/c^2)}{(1+vV_x/c^2)^2}
\end{aligned} \tag{2}$$

Since the second term on the right is negative, it immediately follows that the right side is less than c^2. Also the second term vanishes, when $v^2 = c^2$ or $\boldsymbol{V}^2 = c^2$ or both, so in all of these cases the resultant speed is that of light.

6. We can form two scalar fields from the field strength tensor $F_{\mu\nu}(x)$;

$$\begin{aligned}
I_1(x) &= \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(c^2\boldsymbol{B}^2-\boldsymbol{E}^2), \qquad I_2(x) = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = -2c\boldsymbol{E}\cdot\boldsymbol{B} \\
I'_{1,2}(x') &= I_{1,2}(x) = I_{1,2}(\Lambda^{-1}x')
\end{aligned}$$

If the fields are uniform in space-time, $I_{1,2}$ are just numbers which have the same values in all inertial frames. Their values control whether it is possible to find certain kinds of special frames.

a) What criteria on uniform static $\boldsymbol{E}, \boldsymbol{B}$ must be met if we are to find a frame in which one or the other or both are zero? Are there any fields which are purely electric in one inertial frame and purely magnetic in another inertial frame?

Solution In a frame where $\boldsymbol{E} = 0$, inspection shows $I_1 > 0$ and $I_2 = 0$. Thus this frame can be reached from a frame with both $\boldsymbol{E}, \boldsymbol{B} \neq 0$ only if $|\boldsymbol{E}| < c|\boldsymbol{B}|$ and $\boldsymbol{E}\cdot\boldsymbol{B} = 0$. In a frame where $\boldsymbol{B} = 0$, inspection shows $I_1 < 0$ and $I_2 = 0$. Thus this frame can be reached from a frame with both $\boldsymbol{E}, \boldsymbol{B} \neq 0$ only if $|\boldsymbol{E}| > c|\boldsymbol{B}|$ and $\boldsymbol{E}\cdot\boldsymbol{B} = 0$. If there were a Lorentz transformation that took a frame with $\boldsymbol{E} = 0$ to one where $\boldsymbol{B} = 0$, then I_1 would have to be both greater and less than zero, so it must be that $I_1 = I_2 = 0$ implying $|\boldsymbol{E}| = c|\boldsymbol{B}|$ in all frames. This means that $\boldsymbol{E} = \boldsymbol{B} = 0$ in both frames. Then $F_{\mu\nu} = 0$ in both frames. But a tensor that is zero in one frame is zero in all frames so no nonzero fields have this property.

b) Suppose in the lab frame the angle between static uniform electric and magnetic fields $\boldsymbol{E}, \boldsymbol{B}$ with $c|\boldsymbol{B}| > |\boldsymbol{E}|$ is $\theta \neq 0, \pi/2$. (For definiteness take axes so $\boldsymbol{E}$ is parallel to the x-axis and $\boldsymbol{B}$ lies in the xy-plane.) Determine an inertial frame in which $\boldsymbol{E}$ and $\boldsymbol{B}$ are parallel.

Solution Consider a boost in the z-direction:

$$E'_x = \gamma(E - vB\sin\theta), \quad E'_y = \gamma vB\cos\theta, \quad B'_x = \gamma B\cos\theta, \quad B'_y = \gamma(B\sin\theta - vE/c^2) \tag{3}$$

$\boldsymbol{E}', \boldsymbol{B}'$ will be parallel if their components are in the same ratio, i.e.

$$\frac{vB\cos\theta}{E - vB\sin\theta} = \frac{B\sin\theta - vE/c^2}{B\cos\theta} \tag{4}$$

which leads to a quadratic equation for v with solutions

$$v_\pm = c\frac{B^2 + E^2/c^2 \pm \sqrt{(B^2 + E^2/c^2)^2 - 4(EB/c)^2\sin^2\theta}}{(2EB/c)\sin\theta} \tag{5}$$

The condition $v < c$ dictates that the $v = v_-$.

c) What are the fields in the frame determined in b) in the case θ is very small, and in the case θ is very close to $\pi/2$?

Solution As $\theta \to 0$ $v \sim c^2EB\theta/(c^2B^2 + E^2) = O(\theta)$ and $\gamma = 1 + O(\theta^2)$. Thus $E'_x = E + O(\theta^2), E'_y = O(\theta), B'_x = B + O(\theta^2), B'_y = O(\theta)$.
For $\theta \to \pi/2$, $v \to (c^2B^2 + E^2 - |c^2B^2 - E^2|)/2EB = E/B$ when $c^2B^2 > E^2$. Then $\gamma \to cB/\sqrt{c^2B^2 - E^2}$, $E'_{x,y}, B'_x \to 0$ and $B'_y \to \sqrt{B^2 - E^2/c^2}$. It is the frame where $\boldsymbol{E}' = 0$.

7. Consider a point charge q moving along the positive x-axis at velocity v. In the primed frame moving with the particle, the electric field is just the static Coulomb field and the magnetic field is zero.

a) By applying the Lorentz transformation to these fields in the primed frame, show that in the unprimed frame the fields are

$$\boldsymbol{E} = \frac{q(\boldsymbol{r} - vt\hat{x})}{4\pi\epsilon_0\gamma^2[(x - vt)^2 + (y^2 + z^2)(1 - v^2/c^2)]^{3/2}}, \qquad \boldsymbol{B} = \frac{v}{c^2}\hat{x} \times \boldsymbol{E}$$

Solution: The components of the fields parallel to the boost are the x components and are invariant:

$$\begin{aligned}
E^x(x,y,z,t) &= E'^x(x',y,z,t') = \frac{q}{4\pi\epsilon_0}\frac{x'}{(x'^2+y^2+z^2)^{3/2}} = \frac{q}{4\pi\epsilon_0}\frac{\gamma(x-vt)x}{(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}} \\
&= \frac{q(x-vt)}{4\pi\epsilon_0\gamma^2[(x-vt)^2+(y^2+z^2)(1-v^2/c^2)]^{3/2}} \\
B^x &= B'^x = 0
\end{aligned} \tag{6}$$

Meanwhile the perpendicular components transform as

$$\begin{aligned}
E^{y,z} &= \gamma(E'^{y,z} - (v\hat{x}\times \boldsymbol{B}')^{y,z}) = \gamma E'^{y,z} \\
&= \frac{q(y,z))}{4\pi\epsilon_0\gamma^2[(x-vt)^2+(y^2+z^2)(1-v^2/c^2)]^{3/2}} \\
B^{y,z} &= \gamma(B'^{y,z} + ((v/c^2)\hat{x}\times \boldsymbol{E}')^{y,z})) = \frac{v}{c^2}(\hat{x}\times\gamma(E'^y\hat{y}+E'^z\hat{z})^{y,z}) \\
&= \frac{v}{c^2}(\hat{x}\times\boldsymbol{E})^{y,z}
\end{aligned} \tag{7}$$

Comparing to the desired results we find complete agreement.

b) Notice that $\boldsymbol{E}$ is directed radially from the present location of the charge, but its strength is not isotropic. Compare the electric field at a large distance R from the charge along the x axis to the field at the same large distance R from the charge along the y (or z) axis. What happens to their ratio for v very close to the speed of light?

Solution: Inspecting the coordinate dependence of the denominator, we see that

$$\begin{aligned}
[(x-vt)^2+(y^2+z^2)(1-v^2/c^2)]^{3/2} &\rightarrow \begin{cases} R^3 & \text{for } x-vt=R, y=z=0 \\ R^3/\gamma^3 & \text{for } x=vt, y=R, z=0 \end{cases} \\
\frac{|\boldsymbol{E}(R+vt,0,0)|}{|\boldsymbol{E}(vt,R,0)|} &= \frac{1}{\gamma^3} = \left(1-\frac{v^2}{c^2}\right)^{3/2}
\end{aligned} \tag{8}$$

As $v\rightarrow c$ the fields are concentrated more and more into the plane perpendicular to the charge's velocity. (See Figure 11.9 on page 561 of Jackson).

8. J, Problem 11.18. Please do this problem in SI units. Then the quoted electric field in part a) should have the right side multiplied by $1/4\pi\epsilon_0$, and the quoted magnetic field should be multiplied by $1/4\pi c\epsilon_0$. The potentials quoted in part c) will acquire similar factors.

Solution

a) The fields in SI units are given in the solution to the previous problem, where the x-components of that problem are the z- components in this problem. When $z \neq vt$, the factors of $1/\gamma^2$ make all components of the electric and magnetic fields vanish when $v \to c$. However when $z = vt$, the perpendicular components are now proportional to γ and therefore blow up as $v \to c$, while the z components still vanish in that limit (because they are proportional to $x - vt$. Thus the perpendicular components have the character of a delta function $\delta(z - ct)$ in that limit. To find the coefficient we integrate over z, making the change of variable $z = \sqrt{\boldsymbol{r}_\perp^2(1 - v^2/c^2)}\tan\theta$:

$$\begin{aligned}\int_{-\infty}^{\infty} dz \frac{1}{4\pi\epsilon_0\gamma^2[z^2 + \boldsymbol{r}_\perp^2(1 - v^2/c^2)]^{3/2}} &= \int_{-\pi/2}^{\pi/2} d\theta \sec^2\theta \frac{\sqrt{\boldsymbol{r}_\perp^2(1 - v^2/c^2)}^{-2}}{4\pi\epsilon_0\gamma^2\sec^3\theta} \\ &= \frac{1}{4\pi\epsilon_0\boldsymbol{r}_\perp^2}\int_{-\pi/2}^{\pi/2} d\theta \cos\theta = 2\frac{1}{4\pi\epsilon_0\boldsymbol{r}_\perp^2} \end{aligned} \tag{9}$$

which means that in the limit $v \to c$ this factor behaves as

$$\frac{1}{4\pi\epsilon_0\gamma^2[(z - vt)^2 + \boldsymbol{r}_\perp^2(1 - v^2/c^2)]^{3/2}} \to 2\frac{1}{4\pi\epsilon_0\boldsymbol{r}_\perp^2}\delta(z - ct). \tag{10}$$

It then follows that the fields go to

$$\boldsymbol{E} = \frac{2q}{4\pi\epsilon_0}\frac{\boldsymbol{r}_\perp}{\boldsymbol{r}_\perp^2}\delta(z - ct), \qquad \boldsymbol{B} = \frac{1}{c}\hat{\boldsymbol{v}} \times \boldsymbol{E} \tag{11}$$

which was to be shown.

b) We compute

$$\nabla \cdot \boldsymbol{E} = \frac{2q}{4\pi\epsilon_0}\delta(z - ct)\nabla_\perp \cdot \frac{\boldsymbol{r}_\perp}{\boldsymbol{r}_\perp^2} = \frac{q}{\epsilon_0}\delta(z - ct)\delta(\boldsymbol{r}_\perp) \tag{12}$$

where we used that fact that $\boldsymbol{r}_\perp/2\pi\epsilon_0\boldsymbol{r}_\perp^2$ is the two dimensional electric field of a point charge and so has a divergence of $\delta(\boldsymbol{r}_\perp)/\epsilon_0$. It follows that the charge density is $\rho = q\delta(z - ct)\delta(\boldsymbol{r}_\perp) = J^0/c$. To identify the current density $\boldsymbol{J}$ we take the curl of $\boldsymbol{B}$:

$$\begin{aligned}\frac{1}{c}\nabla \times (\hat{\boldsymbol{v}} \times \boldsymbol{E}) &= \frac{1}{c}\hat{\boldsymbol{v}}\nabla \cdot \boldsymbol{E} - \frac{1}{c}(\hat{\boldsymbol{v}} \cdot \nabla)\boldsymbol{E} = \frac{1}{c}\hat{\boldsymbol{v}}\frac{\rho}{\epsilon_0} - \frac{1}{c}\frac{\partial}{\partial z}\boldsymbol{E} \\ &= \mu_0 c\hat{\boldsymbol{v}}\rho + \frac{1}{c^2}\frac{\partial}{\partial t}\boldsymbol{E} = \mu_0 c\hat{\boldsymbol{v}}\rho + \mu_0\epsilon_0\frac{\partial \boldsymbol{E}}{\partial t}\end{aligned} \tag{13}$$

Comparing to the Ampere-Maxwell Equation we see that $\boldsymbol{J} = c\hat{\boldsymbol{v}}\rho$.

c) With $\phi = -(q/2\pi\epsilon_0)\ln|\boldsymbol{r}_\perp| = A^z c$, and $\boldsymbol{A}_\perp = 0$, we check that

$$\begin{aligned}E^z &= -\frac{\partial\phi}{\partial z} - \frac{\partial A^z}{\partial t} = -\left(c\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)A^z = 0 \\ \boldsymbol{E}_\perp &= -\nabla_\perp\phi = \frac{q}{2\pi\epsilon_0}\frac{\boldsymbol{r}_\perp}{\boldsymbol{r}_\perp^2}\delta(z - ct) \\ \boldsymbol{B} &= \nabla \times \boldsymbol{A} = \nabla A^z \times \hat{z} = -\frac{1}{c}\boldsymbol{E} \times \hat{z} = \frac{1}{c}\hat{\boldsymbol{v}} \times \boldsymbol{E}\end{aligned} \tag{14}$$

Alternatively we can take $\phi = A^z c = 0$ and $\boldsymbol{A}_\perp = \frac{1}{c}\Theta(z-ct)(q/2\pi\epsilon_0)(\boldsymbol{r}_\perp/\boldsymbol{r}_\perp^2)$, and check that

$$\begin{aligned} E^z &= 0, \qquad \boldsymbol{E}_\perp = -\frac{\partial \boldsymbol{A}}{\partial t} = \Theta'(z-ct)\frac{q}{2\pi\epsilon_0}\frac{\boldsymbol{r}_\perp}{\boldsymbol{r}_\perp^2} = \delta(z-ct)\frac{q}{2\pi\epsilon_0}\frac{\boldsymbol{r}_\perp}{\boldsymbol{r}_\perp^2} \\ \boldsymbol{B} &= \nabla\times\boldsymbol{A}_\perp = \hat{z}\times\frac{\partial \boldsymbol{A}}{\partial z} + \nabla_\perp\times\boldsymbol{A}_\perp = \hat{z}\times\frac{\partial \boldsymbol{A}}{\partial z} = \frac{1}{c}\hat{z}\times\boldsymbol{E} = \frac{1}{c}\hat{\boldsymbol{v}}\times\boldsymbol{E} \qquad (15) \end{aligned}$$

where we used the fact that $\boldsymbol{A}_\perp$ is a transverse gradient, and hence has a zero transverse curl:

$$\boldsymbol{A}_\perp = \nabla_\perp\frac{1}{c}\Theta(z-ct)\frac{q}{2\pi\epsilon_0}\ln|\boldsymbol{r}_\perp| \equiv -\nabla_\perp\chi \qquad (16)$$

The gauge transformation χ will set this $\boldsymbol{A}_\perp = 0$ and set A^z, ϕ to the values in the first case:

$$\begin{aligned} A^z &\rightarrow 0 + \frac{\partial\chi}{\partial z} = -\frac{1}{c}\delta(z-ct)\frac{q}{2\pi\epsilon_0}\ln|\boldsymbol{r}_\perp| \\ \phi &\rightarrow 0 - \frac{\partial\chi}{\partial t} = -\delta(z-ct)\frac{q}{2\pi\epsilon_0}\ln|\boldsymbol{r}_\perp| \qquad (17) \end{aligned}$$