Electromagnetic Theory II

Solution Set 3

Due: 3 February 2021

9. Consider the Lorentz transformation properties of the 4-vector current field $J^\mu = (c\rho, \boldsymbol{J})$. Let the primed frame move with velocity $\boldsymbol{v} = v\hat{x}$, parallel to the x-axis, relative to the unprimed frame. Be sure to use SI units, and to take account of the conditions on the moments of $\boldsymbol{J}$ following from $\nabla \cdot \boldsymbol{J} = 0$!

a) Write out the Lorentz transformation which expresses $J^\mu(x)$ in terms of $J'(x') = J'(\Lambda x)$

Solution

$$\rho(\boldsymbol{x}, t) = \gamma\left(\rho'(\boldsymbol{x}') + \frac{v}{c^2}J'_x(\boldsymbol{x}')\right), \quad J_x(\boldsymbol{x}, t) = \gamma\left(J'_x(\boldsymbol{x}') + v\rho'(\boldsymbol{x}')\right), \quad J_{y,z}(\boldsymbol{x}, t) = J'_{y,z}(\boldsymbol{x}') \quad (1)$$

where $x' = \gamma(x - vt)$, $y' = y$, and $z' = z$. The densities in the unprimed frame are time dependent because they are moving.

b) Suppose the current density $\boldsymbol{J}' = 0$ and the charge density ρ' in the primed frame is static, has 0 total charge, but has a non zero electric dipole moment $\boldsymbol{p}'$. By working out the charge and current densities in the unprimed frame, find the electric ($\boldsymbol{p}$) and magnetic ($\boldsymbol{m}$) dipole moments in the unprimed frame. Show the limiting cases of $v << c$, where you should find $\boldsymbol{m} \approx \boldsymbol{p} \times \boldsymbol{v}/2$.

Solution The integrals for the dipole moments in the unprimed frame are done at constant t. so we use the relation $x' = \gamma(x - vt)$ or $x = vt + x'/\gamma$ to change variables

$$\begin{aligned}\boldsymbol{p} &= \gamma\int dxdydz(x, y, z)\rho'(x', y', z') = \int dx'dy'dz'(vt + x'/\gamma, y', z')\rho'(x', y', z') \\ &= (p'_x/\gamma, p'_y, p'_z) \approx \boldsymbol{p}' \end{aligned} \quad (2)$$

The vt term drops out because the distribution has zero total charge. The magnetic moment reads

$$\begin{aligned}\boldsymbol{m} &= \frac{1}{2}\int d^3x\boldsymbol{r} \times \boldsymbol{J} = \frac{\gamma v}{2}\int d^3x(-y\hat{z} + z\hat{y})\rho'(x', y', z') = \frac{v}{2}\int d^3x'(-y'\hat{z} + z'\hat{y})\rho' \\ &= \frac{v}{2}(0, p'_z, -p'_y) = \frac{1}{2}\boldsymbol{p}' \times \boldsymbol{v} \approx \frac{1}{2}\boldsymbol{p} \times \boldsymbol{v} \end{aligned} \quad (3)$$

c) Now do the case where the charge density $\rho' = 0$ and the current density $\boldsymbol{J}'$ is static and such as to produce a magnetic moment $\boldsymbol{m}'$. Find the electric and magnetic moments and their nonrelativistic limits, where you should find $\boldsymbol{p} \approx \boldsymbol{v} \times \boldsymbol{m}/c^2$.

Solution

$$\begin{aligned}
\boldsymbol{m} &= \frac{1}{2}\int \frac{d^3x'}{\gamma}(yJ'_z - zJ'_y, z\gamma J'_x - (x'/\gamma + vt)J'_z, (x'/\gamma + vt)J'_y - \gamma y J'_x) \\
&= \big(m'_x/\gamma, m'_y(1+\gamma)/2\gamma, m'_z(1+\gamma)/2\gamma\big) \approx \boldsymbol{m}' \qquad (4)
\end{aligned}$$

Here the vt terms drop out because of the identity $\int d^3x' \boldsymbol{J}' = 0$, following from $\nabla' \cdot \boldsymbol{J}' = 0$. And the y, z components were simplified using the identities $\int d^3x'(x'^i J'^j + x'^j J'^i) = 0$. The electric dipole moment is (making use of the same identities)

$$\begin{aligned}
\boldsymbol{p} &= \int \frac{d^3x'}{\gamma}(x'/\gamma + vt, y', z')\frac{\gamma v}{c^2}J'_x = \frac{v}{c^2}\int d^3x'(0, y', z')J'_x \\
&= \frac{v}{c^2}(0, -m'_z, m'_y) = \boldsymbol{v}\times\boldsymbol{m}'/c^2 \approx \boldsymbol{v}\times\boldsymbol{m}/c^2 \qquad (5)
\end{aligned}$$

10. J, Problem 11.23

Solution

a),b)Evaluate the invariant $(p_1 + p_2)^2$ in the two frames:

$$\begin{aligned}
(p_1 + p_2)^2 &= p_1^2 + p_2^2 + 2p_1 \cdot p_2 = -(m_1^2 + m_2^2)c^2 - 2E_{\rm Lab}m_2 \\
(p_1 + p_2)^2 &= -\frac{(E_1^{\rm CM} + E_2^{\rm CM})^2}{c^2} \equiv -\frac{W^2}{c^2} \\
W^2 &= (m_1^2 + m_2^2)c^4 + 2E_{\rm Lab}m_2c^2 \qquad (6)
\end{aligned}$$

The CM frame is a boost in the direction of $\boldsymbol{p}$ which sets the toral momentum in the new frame to zero:

$$0 = \boldsymbol{P}' = \gamma(\boldsymbol{p}_{\rm Lab} - \boldsymbol{v}(m_2c^2 + E_{\rm Lab})) \qquad (7)$$

Implying that the boost velocity is $\boldsymbol{v} = \boldsymbol{p}c^2/E = \boldsymbol{p}_{\rm Lab}c^2/(m_2c^2 + E_{\rm Lab})$. Then the boost of particle 1 to the CM frame reads

$$\boldsymbol{p}' = \gamma\left(\boldsymbol{p}_{\rm Lab} - \frac{\boldsymbol{p}_{\rm Lab}}{m_2c^2 + E_{\rm Lab}}E_{\rm Lab}\right) = \frac{\gamma m_2c^2}{m_2c^2 + E_{\rm Lab}}\boldsymbol{p}_{\rm Lab} = \frac{m_2c^2}{W}\boldsymbol{p}_{\rm Lab} \qquad (8)$$

The last equality follows since the total momentum in the CM frame is zero, so the total system energy in the Lab is $m_2c^2 + E_{\rm Lab} = \gamma W$.

c) In the NR limit $p_{\rm Lab} \ll m_{1,2}c^2$ and $E_{\rm Lab} \approx m_1c^2 + \boldsymbol{p}_{\rm Lab}^2/2m_1$. Then $\boldsymbol{\beta} \approx \boldsymbol{p}_{\rm Lab}/(m_1 + m_2)$,

$$\begin{aligned}
W &= \sqrt{(m_1^2 + m_2^2)c^4 + 2E_{\rm Lab}m_2c^2} \approx \sqrt{(m_1 + m_2)^2c^4 + \frac{\boldsymbol{p}_{\rm Lab}^2 m_2c^2}{m_1}} \\
&\approx (m_1 + m_2)c^2 + \frac{\boldsymbol{p}_{\rm Lab}^2 m_2}{2m_1(m_1 + m_2)} \\
\boldsymbol{p}' &\approx \frac{m_2}{m_1 + m_2}\boldsymbol{p}_{\rm Lab} \qquad (9)
\end{aligned}$$

11. J, Problem 11.25

Solution:

a)

$$\begin{aligned}
W^2 &= -c^2(p_1+p_2)^2 = -c^2(p_1^2+p_2^2+2p_1\cdot p_2) = (m_1^2+m_2^2)c^4 + 2c^2(E_1E_2 - \boldsymbol{p}_1\cdot\boldsymbol{p}_2) \\
&= (m_1^2+m_2^2)c^4 + 2c^2p_1p_2\left(\sqrt{1+\frac{m_1^2c^2}{p_1^2}}\sqrt{1+\frac{m_2^2c^2}{p_2^2}}+\cos\theta\right) \\
&\sim 2p_1p_2c^2(1+\cos\theta) + m_1^2c^4(1+p_2/p_1) + m_2^2c^4(1+p_1/p_2) \\
&\sim 4p_1p_2c^2\cos^2(\theta/2) + (p_1+p_2)c^4\left(\frac{m_1^2}{p_1}+\frac{m_2^2}{p_2}\right) \qquad (10)
\end{aligned}$$

b) The system as a whole is at rest in the center of mass, so we need the velocity of the system as a whole. For any system this is

$$\boldsymbol{v} = \frac{\boldsymbol{p}_{total}c^2}{E_{total}} = \frac{\boldsymbol{p}_1+\boldsymbol{p}_2}{E_1+E_2}c^2 \qquad (11)$$

From the figure

$$\boldsymbol{p}_1+\boldsymbol{p}_2 = (p_1-p_2)\hat{x}\cos\theta/2 + (p_1+p_2)\hat{y}\sin\theta/2 \qquad (12)$$

from which we see that

$$\begin{aligned}
\tan\alpha &= \frac{p_1+p_2}{p_1-p_2}\tan\frac{\theta}{2} \\
\boldsymbol{p}_1+\boldsymbol{p}_2 &= (p_1+p_2)\tan\frac{\theta}{2}\cot\alpha\hat{x}\cos\frac{\theta}{2} + (p_1+p_2)\hat{y}\sin\frac{\theta}{2} = (p_1+p_2)\frac{\sin\theta/2}{\sin\alpha}(\hat{x}\cos\alpha+\hat{y}\sin\alpha)
\end{aligned}$$

Thus the magnitude $v/c = c(p_1+p_2)\sin(\theta/2)/(E_1+E_2)\sin\alpha$

c) The conditions of 11.23 b) are that $p_2 = 0$ and hence $\alpha = \theta/2$. Then

$$\boldsymbol{\beta} = \frac{p_1c}{E_1+m_2c^2}(\hat{x}\cos\frac{\theta}{2}+\hat{y}\sin\frac{\theta}{2}) = \frac{\boldsymbol{p}_1c}{E_1+m_2c^2} \qquad (13)$$

in accordance with 11.23b

d) For $p_1 = p_2$ the direction of the β_{cm} is $\alpha = \pi/2$ and its value is approximately $\sin 10^\circ \approx 0.17$. the corresponding $\gamma \approx 1+\beta^2/2 \approx 1.015$. Let us consider the effect of this boost on two pions produced with equal and opposite momenta in the center of mass. If the direction of those momenta are perpendicular to the boost, then in the Lab the perpendicular components will be the same, but they will each acquire a component of momentum in the boost direction of size $\gamma\beta\sqrt{p^2+m_\pi^2c^2} \approx \sin 10^\circ p$. In this worst case they will fail to be collinear by 20°.

12. As we discussed in class, solving the motion of a particle in uniform electric and magnetic fields is straightforward, but tedious. The equations of motion

$$\frac{dU_\mu}{d\tau} = \frac{q}{mc} F_{\mu\nu} U^\nu \qquad (14)$$

form a set of four coupled first order differential equations with constant coefficients for the four quantities $U_{0,1,2,3}$, which can be solved by first assuming all components have the same exponential time dependence $U_\mu = K_\mu e^{r\tau}$ so the differential equations reduce to a set of 4 linear algebraic equations for the K^r_μ associated with each r, and the consistency conditions $\det(F - rI) = 0$ that restrict the values of r to the four solutions of a quartic equation. The general solution can be constructed as a linear combination of these 4 special solutions.

a) Write out these equations in the coordinate system where $\boldsymbol{E} = E\hat{x}$ and $\boldsymbol{B} = B(\hat{x}\cos\theta + \hat{y}\sin\theta)$. Here θ is the angle between $\boldsymbol{B}$ and $\boldsymbol{E}$.

Solution: The matrix $F^\mu_{\ \nu}$ reads

$$\begin{aligned}
F &= \begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & -cB\sin\theta \\ 0 & 0 & 0 & cB\cos\theta \\ 0 & cB\sin\theta & -Bc\cos\theta & 0 \end{pmatrix} \\
\frac{dU^0}{d\tau} &= \frac{qE}{mc}U^1, \qquad \frac{dU^1}{d\tau} = \frac{qE}{mc}U^0 - \frac{qB}{m}\sin\theta U^3 \\
\frac{dU^2}{d\tau} &= \frac{qB}{m}\cos\theta U^3, \qquad \frac{dU^3}{d\tau} = \frac{qB}{m}\sin\theta U^1 - \frac{qB}{m}\cos\theta U^2
\end{aligned} \qquad (15)$$

b) In view of the result of problem 6b) in Set 2, we can find an inertial frame where $\theta = 0$. Thus we actually lose no information by solving only this case. So for $\theta = 0$ find the general solution to the equations of part a) for $U_{0,1,2,3}(\tau)$ and then for $x_{0,1,2,3}$ by integrating $dx_\mu/d\tau = U_\mu(\tau)$. Remember that U^μ is subject to the constraint $U_\mu U^\mu = -c^2$, which restricts the overall scale of the coefficients in this linear combination. For simplicity, set the initial conditions: $U^1(0) = 0$ and $x^0(0) = 0$.

Solution: For $\theta = 0$ the pair of equations for $U^{0,1}$ decouple from the pair for $U^{2,3}$. These can be solved by direct substitution:

$$\begin{aligned}
U^1 &= \frac{mc}{qE}\frac{dU^0}{d\tau}, \qquad \frac{d^2U^0}{d\tau^2} = \frac{q^2E^2}{m^2c^2}U^0 \\
U^0 &= Ae^{qE\tau/mc} + De^{-qE\tau/mc}, \qquad U^1 = Ae^{qE\tau/mc} - De^{-qE\tau/mc} \\
U^3 &= \frac{m}{qB}\frac{dU^2}{d\tau}, \qquad \frac{d^2U^2}{d\tau^2} = -\frac{q^2B^2}{m^2}U^2 \\
U^2 &= Ce^{iqB\tau/m} + C^*e^{-iqB\tau/m}, \qquad U^3 = iCe^{iqB\tau/m} - iC^*e^{-iqB\tau/m} \\
-c^2 &= U_\mu U^\mu = 4CC^* - 4AD, \qquad D = \frac{|C|^2 + c^2/4}{A}
\end{aligned}$$

Then we integrate once more to obtain $x^\mu(\tau)$:

$$\begin{aligned} x^0(\tau) &= X^0 + \frac{mc}{qE}U^1, \qquad x^1(\tau) = X^1 + \frac{mc}{qE}U^0 \\ x^2(\tau) &= X^2 - \frac{m}{qB}U^3, \qquad x^3(\tau) = X^3 + \frac{m}{qB}U^2 \end{aligned}$$

A, C, X^μ are integration constants, parametrized by 7 real numbers (since C is complex). We can reduce the number of parameters by choice of the zeroes of τ and x^0. First notice that from the constraint D and A have the same sign. Thus we can choose τ so that $U^1(0) = 0$, i.e. $D = A$, with $A^2 = |C|^2 + c^2/4$. Then we can choose $X^0 = 0$, so that $x^0(0) = 0$. With these choices, we then have:

$$U^0 = 2A\cosh\frac{qE\tau}{mc}, \qquad U^1 = 2A\sinh\frac{qE\tau}{mc}, \qquad x^0(\tau) = \frac{2Amc}{qE}\sinh\frac{qE\tau}{mc} \quad (16)$$

and the motion is parametrized by the 5 parameters $C, \boldsymbol{X}$.

c) Specialize your solutions to zero magnetic field ($B = 0$), assuming $x^2 = x^3 = 0$, and recover the solutions for uniform electric field discussed in class.

Solution: With the chosen initial conditions

$$x^0(\tau) = 2A\frac{mc}{qE}\sinh\frac{qE\tau}{mc}, \qquad x^1(\tau) = X^1 + 2A\frac{mc}{qE}\cosh\frac{qE\tau}{mc}$$

Using the identity $\cosh = \sqrt{1+\sinh^2}$, we can express x^1 as a function of $x^0 = ct$:

$$x^1(t) = X^1 + 2A\frac{mc}{qE}\sqrt{1+\frac{q^2E^2t^2}{4m^2A^2}} = X^1 + \frac{c}{qE}\sqrt{4m^2A^2 + q^2E^2t^2} \quad (17)$$

With $A = D$ and $C - 0$, the constraint gives $4A^2 - c^2$, wo we have finally

$$x^1(t) = X^1 + \frac{c}{qE}\sqrt{m^2c^2 + q^2E^2t^2} \quad (18)$$

which agrees with the result (486) in our lecture notes.