Electromagnetic Theory II

Solution Set 4

Due: 10 February 2021

13. J, Problem 12.7 **in SI units**. Note that in SI units, the inequalities mentioned in parts b) and c) are $p > qBa$ and $p < qBa/2$ respectively. Also, in these units, the electromagnetic momentum density is just $\boldsymbol{g} = \boldsymbol{D} \times \boldsymbol{B} \to \epsilon_0 \boldsymbol{E} \times \boldsymbol{B}$ in empty space.

Solution:

a) The electric $\boldsymbol{D}$ field of a point charge at a point (x_0, y_0, z_0) is given by

$$\boldsymbol{D}(\boldsymbol{r}) = \frac{q}{4\pi}\frac{(x-x_0)\hat{x} + (y-y_0)\hat{y} + (z-z_0)\hat{z}}{((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2)^{3/2}}$$

Since the magnetic field is zero outside the two planes, so is the momentum density. Then the total momentum is given by

$$\begin{aligned} \boldsymbol{G} &= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \int_0^a dx \boldsymbol{D} \times \boldsymbol{B} \\ &= \frac{q\hat{x}}{4\pi} \times [-B\hat{z}] \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \int_0^a dx \frac{(x-x_0)}{((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2)^{3/2}} \end{aligned} \quad (1)$$

The y and z components of $\boldsymbol{D}$ integrate to zero because either the y or z integrand is odd. In what remains the $dydz$ integral is best done in polar coordinates with origin at (y_0, z_0):

$$\begin{aligned} \boldsymbol{G} &= \frac{qB\hat{y}}{4\pi} 2\pi \int_0^a dx \int_0^{\infty} \rho d\rho \frac{x-x_0}{((x-x_0)^2 + \rho^2)^{3/2}} = \frac{qB\hat{y}}{2} \int_0^a dx \frac{x-x_0}{|x-x_0|} \\ &= \begin{cases} \frac{1}{2} qBa\hat{y} & x_0 < 0 \\ \frac{1}{2} qB(a-2x_0)\hat{y} & 0 < x_0 < a \\ -\frac{1}{2} qBa\hat{y} & x_0 > a \end{cases} \end{aligned} \quad (2)$$

where we used the fact that the dx integrand is simply the sign of $x - x_0$.

b) Because the magnetic field is in the negative z direction a positively charged particle will follow a counterclockwise circle in the xy-plane looking down on it. The radius of the circle is $R = p/qB$ where p is the constant magnitude of the momentum. For $p > qBa$, $R > a$ so a particle entering the region normally will exit it after traveling an arc of angle θ, with $\sin\theta = a/R$. From the geometry, we can see that the exiting particle has components of momentum $p_x = p\cos\theta = p\sqrt{R^2 - a^2}/R$, $p_y = p\sin\theta = pa/R = qBa$. Initially $p_x = qBR$, $p_y = 0$. Initialy $G_x = 0$, $G_y = qBa/2$ and finally $G_x = 0$, $G_y = -qBa/2$. Thus we see that $p_y + G_y$ is the same intially and finally, whereas $p_x + G_x$ is not. This is reasonable since there is translational invariance in the y direction, but not in the x direction.

c) If $p < qBa/2$ the circular path can lie completely in the region of non-zero magnetic field. Its circular trajectory can then be described by

$$x(t) = R\cos\omega_B t, \qquad y(t) = R\sin\omega_B t, \qquad p_x = -qBR\sin\omega_B t, \qquad p_y = qBR\cos\omega_B t$$

We then find

$$\begin{aligned} p_y + G_y &= qBR\cos\omega_B t + \frac{qBa}{2} - qBx(t) = \frac{qBa}{2} = \text{constant} \\ p_x + G_x &= p_x = -qBR\sin\omega_B t \end{aligned} \tag{3}$$

so the y component of total momentum is conserved throughout the motion, whereas the x-component is not. This is in accord with translational invariance in the y-direction and broken translational invariance in the x-direction.

Optional Discussion: At a more formal level the conservation laws of this problem can derived by writing

$$\frac{d\boldsymbol{p}}{dt} = q\boldsymbol{v}\times\boldsymbol{B} = \int d^3x \boldsymbol{J}\times\boldsymbol{B} = \int d^3x\left(\nabla\times\boldsymbol{H}_i - \frac{\partial\boldsymbol{D}_i}{\partial t}\right)\times\boldsymbol{B}$$

where $\boldsymbol{H}_i, \boldsymbol{D}_i$ are the magnetic and electric fields induced by the moving charge. Then

$$\begin{aligned} \frac{d}{dt}\left(\boldsymbol{p} + \int d^3x\boldsymbol{D}_i\times\boldsymbol{B}\right) &= \int d^3x(\nabla\times\boldsymbol{H}_i)\times\boldsymbol{B} = \int dydz(\hat{x}\times\boldsymbol{H}_i)\times(-B\hat{z})\Big|_0^a \\ &= B\hat{x}\int dydz[H_i^z(a,y,z,t) - H_i^z(0,y,z,t)] \end{aligned}$$

which shows the conservation of the y-component of the total momentum, but not the x-component.

14. J, Problem 12.9

Solution:

a) The magnetic field of a dipole points in the direction $3\boldsymbol{m}\cdot\boldsymbol{r}\boldsymbol{r} - r^2\boldsymbol{m}$. The tangent of a field line passing a point $\boldsymbol{r}$ must be parallel to this direction. Parametrizing a field line in the xz-plane $\boldsymbol{r}(\theta) = r(\theta)(\sin\theta, 0, \cos\theta)$, its tangent points in the direction $d\boldsymbol{r}/d\theta = r'(\theta)(\sin\theta, 0, \cos\theta) + r(\theta)(\cos\theta, 0, -\sin\theta)$, which must be parallel to $3r^2(\theta)\cos\theta(\sin\theta, 0\cos\theta) - r^2(\theta)(0,0,1)$,which implies

$$\frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{3\cos\theta\sin\theta}{3\cos^2\theta - 1} \tag{4}$$

For $r = r_0\sin^2\theta$ we easily compute $r'/r = 2\cot\theta$, so the left side of this equation would be

$$\frac{2\cos\theta + \cos\theta}{2\cos\theta\cot\theta - \sin\theta} = \frac{3\cos\theta\sin\theta}{2\cos^2\theta - \sin^2\theta} = \frac{3\cos\theta\sin\theta}{3\cos^2\theta - 1} \tag{5}$$

Thus the parallel condition is met and $r(\theta) = r_0\sin^2\theta$ describes a field line in the field of a dipole. field.

b) The drift velocity is given by the formula

$$\boldsymbol{v}_{drift} = -\frac{\nabla_\perp B \times \boldsymbol{B}}{2B^2} a^2 \omega_B \tag{6}$$

At the equator the $\boldsymbol{B} = \hat{z}B(R)$ where $B(R) \propto 1/R^3$, so $\nabla_\perp B \propto -3\hat{\rho}/R^4$. Then

$$\begin{aligned} \boldsymbol{v}_{drift} &= R\frac{d\varphi}{dt}\hat{\varphi} = 3\hat{\rho} \times \hat{z}\frac{a^2\omega_B}{2R} = -\frac{3}{2}\hat{\varphi}\frac{a^2\omega_B}{R} \\ \varphi(t) &= \varphi_0 - \frac{3}{2}\frac{a^2}{R^2}\omega_B t \end{aligned} \tag{7}$$

c) On the field line $r = R\sin^2\theta$ the magnitude of $\boldsymbol{B}$ is

$$\begin{aligned} B(\theta) &= \frac{\mu_0}{4\pi}M\frac{\sqrt{3\cos^2\theta + 1}}{R^3\sin^6\theta} = \frac{\mu_0}{4\pi}M\frac{\sqrt{3\cos^2\theta+1}}{R^3(1-\cos^2\theta)^3} \approx \frac{M\mu_0}{4\pi R^3}\left(1 + \frac{9}{2}\cos^2\theta\right) \\ \frac{B(\theta)}{B(\pi/2)} &\approx 1 + \frac{9}{2}\cos^2\theta = 1 + \frac{9}{2R^2}z^2 \end{aligned} \tag{8}$$

The longitudinal motion for small z is then governed by the harmonic oscillator potential

$$V(z) = \frac{1}{2}m(a\omega_B)^2\frac{9z^2}{2R^2} \tag{9}$$

from which we read off the oscillator frequency $\Omega = 3(a/R)\omega_B/\sqrt{2}$. The change in longitude over a cycle of oscillation (period $= 2\pi/\Omega$) is

$$\Delta\varphi = -2\pi\frac{3}{2}\frac{a}{R}\frac{\sqrt{2}}{3} = \pi\sqrt{2}\frac{a}{R} \tag{10}$$

d) At $R = 3 \times 10^7$m the equatorial magnetic field is $(8.1/27) \times 10^{-2}$G$= .3 \times 10^{-6}$T. Then $eB/M_e \approx 5.27 \times 10^4$/s. The kinetic enrgy of a relativistic particle is $(\gamma - 1)mc^2$. The rest energy of an electron is .511MeV, so a 10MeV electron has $\gamma_{10} \approx 1 + 19.6 = 20.6$, and a 10keV=.01MeV electron has $\gamma_{.01} \approx 1 + .020 = 1.020$. The corresponding ω_B are $.256 \times 10^4$/s and 5.167×10^4/s respectively. Then $a\omega_B = v = c\sqrt{1 - 1/\gamma^2}$ which is $.999c$ and $.197c$ respectively. The corresponding values of a are 1.17×10^5m and 1.14×10^3m respectively. the ratio a/R is then $.39 \times 10^{-2}$ and $.38 \times 10^{-4}$ respectively. The time to drift once around the earth is $4\pi(R/a)^2/3\omega_B$ which is 108s and 56100s respectively. The oscillation time is $2\pi(R/a)\sqrt{2}/3\omega_B$ which is .297s and 1.51s respectively. Summarizing, we list the results:

	10MeV	10keV
γ	20.6	1.020
ω_B	0.256×10^4/s	5.167×10^4/s
$\dfrac{a}{R}$	0.39×10^{-2}	0.38×10^{-4}
$\dfrac{4\pi}{3\omega_B}\left(\dfrac{R}{a}\right)^2$	108s	56100s
$\dfrac{2\sqrt{2}\pi}{3\omega_B}\dfrac{R}{a}$	0.297s	1.51s

15. Recall the Lagrangian density for scalar electrodynamics

$$\mathcal{L} \;=\; -\frac{\epsilon_0}{4}F_{\mu\nu}F^{\mu\nu} - (\partial_\mu + iQA_\mu)\phi^*(\partial^\mu - iQA^\mu)\phi - U(\phi^*\phi), \qquad F_{\mu\nu} = c(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

a) Derive the canonical energy momentum tensor for this Lagrangian defined as

$$T^{\mu\nu} \;=\; -\sum_i \partial^\mu\psi_i \frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi_i)} + \eta^{\mu\nu}\mathcal{L}$$

where the ψ_i are the 6 independent fields A_ν, ϕ, ϕ^*. Notice that, as we found for the free electodynamics, $T^{\mu\nu}$ is not symmetric in its indices.

Solution We need

$$\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\rho)} \;=\; -\epsilon_0 cF^{\nu\rho}, \quad \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi^*)} = -(\partial^\nu - iQA^\nu)\phi, \quad \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} = -(\partial^\nu + iQA^\nu)\phi^*$$

Let's introduce the shorthand $D^\mu\phi \equiv (\partial^\mu - iQA^\mu)\phi$. D is called the gauge covariant derivative. Then the canonical formula reads

$$\begin{aligned} T^{\mu\nu} &= \epsilon_0 c\partial^\mu A_{\ \rho}F^{\nu\rho} + D^{\nu*}\phi^*\partial^\mu\phi + \partial^\mu\phi^* D^\nu\phi - \eta^{\mu\nu}\left[\frac{\epsilon_0}{4}F_{\rho\sigma}F^{\rho\sigma} + D^*_\rho\phi^* D^\rho\phi + U(\phi^*\phi)\right] \\ &= \epsilon_0 F^\mu_{\ \rho}F^{\nu\rho} + D^{\mu*}\phi^* D^\nu\phi + D^{\nu*}\phi^* D^\mu\phi + \epsilon_0 c\partial_\rho A^\mu F^{\nu\rho} + iQA^\mu\phi D^{\nu*}\phi^* - iQA^\mu\phi^* D^\nu\phi \\ &\quad -\eta^{\mu\nu}\left[\frac{\epsilon_0}{4}F_{\rho\sigma}F^{\rho\sigma} + D^*_\rho\phi^* D^\rho\phi + U(\phi^*\phi)\right] \end{aligned}$$

b) As we did in the free case, find the modification that makes the result of part a) symmetric. Prove that the modification maintains the conservation condition $\partial_\nu T^{\mu\nu} = 0$.

Solution The last three terms on the second line are not symmetric. They read

$$\begin{aligned} \epsilon_0 c\partial_\rho A^\mu F^{\nu\rho} + iQA^\mu\phi D^{\nu*}\phi^* - iQA^\mu\phi^* D^\nu\phi &= \epsilon_0 c\partial_\rho A^\mu F^{\nu\rho} + A^\mu J^\nu \\ &= \epsilon_0 c\partial_\rho(A^\mu F^{\nu\rho}) + A^\mu(J^\nu - \epsilon_0 c\partial_\rho F^{\nu\rho}) \\ &= \epsilon_0 c\partial_\rho(A^\mu F^{\nu\rho}) \end{aligned}$$

by the equations of motion. This asymmetric term is a spatial divergence when $\nu = 0$, so it will not contribute to the total energy momentum $p^\mu \equiv \int d^3x T^{\mu 0}$. Furthermore, it is immediate that $\partial_\nu \partial_\rho (A^\mu F^{\nu\rho}) = 0$ by virtue of the antisymmetry of the field strength tensor. Thus the symmetric tensor obtained by dropping those three terms

$$T^{\mu\nu}_{\text{Sym}} = \epsilon_0 F^\mu{}_\rho F^{\nu\rho} + D^{\mu *}\phi^* D^\nu \phi + D^{\nu *}\phi^* D^\mu \phi - \eta^{\mu\nu}\left[\frac{\epsilon_0}{4} F_{\rho\sigma}F^{\rho\sigma} + D^*_\rho \phi^* D^\rho \phi + U(\phi^*\phi)\right]$$

will be conserved and yield the same values for the energy-momentum. Thus it is a satisfactory alternative form for the energy momentum tensor.

16. We will be using the scalar electrodynamics to discuss superconductivity. In this problem we specialize the energy density to a situation with axially symmetric fields which we will later need to discuss magnetic flux vortices in superconductors. We use cylindrical coordinates ρ, φ, z and assume z-independent fields.

a) Show that the potential $\boldsymbol{A} = \hat{\varphi} A(\rho)$ produces a magnetic field $\boldsymbol{B} = B(\rho)\hat{z}$, and find $B(\rho)$ in terms of $A(\rho)$.

Solution It is simplest to evaluate $\nabla \times \boldsymbol{A}$ in Cartesian coordinates, so we recall that $\hat{\varphi} = -\hat{x}\sin\varphi + \hat{y}\cos\varphi$. Thus $A_x = -yA(\rho)/\rho$, $A_y = xA(\rho)/\rho$, and $A_z = 0$. It follows that $B_x = B_y = 0$ and

$$\begin{aligned} B_z = \partial_x A_y - \partial_y A_x &= A/\rho + x^2 (A/\rho)'/\rho + A/\rho + y^2 (A/\rho)'/\rho \\ &= 2A/\rho + \rho (A/\rho)' = A' + A/\rho = (\rho A)'/\rho \end{aligned}$$

b) For the scalar field we assume a form $\phi = f(\rho)e^{im\varphi}$, with m an integer. Because the fields are z-independent, the total energy is infinite, so it is appropriate to consider instead the energy per unit length $\int \rho d\rho d\varphi T^{00}$. Using the symmetric energy momentum tensor of the previous problem write out the energy per unit length of the magnetic vortex in terms of $A(\rho)$ and $f(\rho)$, expressed as a one dimensional integral over ρ.

Solution Specializing T^{00} to this situation, we first set $\boldsymbol{E} = 0$, $D_0\phi = 0$ to obtain

$$T^{00} = \frac{\epsilon_0 c^2}{2}\boldsymbol{B}^2 + \boldsymbol{D}^*\phi^* \cdot \boldsymbol{D}\phi + U(f^2)$$

Then we need $\nabla\phi = (\hat{\rho} f' + \hat{\varphi}(imf/\rho))e^{im\varphi}$ from which we get

$$\boldsymbol{D}\phi = \left(\hat{\rho} f' + \hat{\varphi} i f\left(\frac{m}{\rho} - QA\right)\right)e^{im\varphi}, \qquad \boldsymbol{D}^*\phi^* \cdot \boldsymbol{D}\phi = f'^2 + \left(\frac{m}{\rho} - QA\right)^2 f^2$$

Putting everything together we find the energy per unit length

$$T = \int \rho d\rho d\varphi T^{00} = 2\pi \int_0^\infty \rho d\rho \left[\frac{\epsilon_0 c^2}{2\rho^2}(\rho A)'^2 + f'^2 + \left(\frac{m}{\rho} - QA\right)^2 f^2 + U(f^2)\right]$$