Electromagnetic Theory II

Problem Set 5

Due: 17 February 2021

17. In problem 16 of set 4 you showed that the energy per unit length of a static and axially symmetric configuration $\boldsymbol{A} = \hat{\varphi}A(\rho)$, $\phi = f(\rho)e^{im\varphi}$ with m an integer, of electromagnetic and scalar fields was

$$T = 2\pi \int_0^\infty d\rho \left[\epsilon_0 c^2 \frac{[(\rho A)']^2}{2\rho} + \rho f'^2 + \rho\left(\frac{m}{\rho} - QA\right)^2 f^2 + \rho U(f^2)\right] \tag{1}$$

The total energy of course would be $\int dz T = \infty$. To describe a superconducting state U must have a minimum for $f \neq 0$. For this problem assume that $U(f^2) = \lambda(f^2 - f_0^2)^2/4$. Static solutions of the equations of motion with axial symmetry are stationary points of T. The lowest energy solution minimizes T. Thus we can use a variational principle to approximate the lowest energy solution for each fixed m. When $m \neq 0$ such a solution will describe a magnetic vortex.

a) Show that in order for T to be finite, the fields must have the behavior $f \to f_0$ and $\rho A \to m/Q$ as $\rho \to \infty$. Evaluate the total magnetic flux $\Phi_B = 2\pi \int_0^\infty \rho d\rho B(\rho)$ for such a vector potential. Notice that it is quantized, and give the fundamental unit.

Solution Since all four terms in T are positive, they all must go to zero as $\rho \to \infty$. The last term vansihes only if $f \to f_0$. Then the third term vanishes only if $A \sim m/Q\rho$ as $\rho \to \infty$. By Stokes theorem

$$\Phi_B = \int dS\boldsymbol{n} \cdot \boldsymbol{B} = \oint d\boldsymbol{l} \cdot \boldsymbol{A} = 2\pi\rho A(\rho) \to 2\pi m/Q \tag{2}$$

as $\rho \to \infty$. It is quantized in units of $2\pi/Q$ because m is an integer.

b) At first glance it would seem that the choices $A = m/Q\rho$ and $f = f_0$ lead to a value of $T = 0$. However, this is too glib. Show that a vector potential of this form implies a magnetic field $\boldsymbol{B} = \Phi_B \delta(x)\delta(y)$. Such a field would contribute $\epsilon_0 c^2 \Phi_B^2 \delta(0)\delta(0)/2 = \infty$ to the energy per unit length. Confirm this conclusion by studying the first term in T for a regularized potential $A = m\rho/Q(\rho^2 + \delta^2)$, showing how it blows up as $\delta \to 0$. Accordingly a finite energy solution must satisfy $A \to 0$ as $\rho \to 0$, and then the third term will only be finite if also $f \to 0$ as $\rho \to 0$.

Solution If $A = m/Q\rho$ exactly then $B(\rho) = [\rho A]'/\rho = 0$ for all $\rho \neq 0$. Nonetheless $\Phi_B = \int dxdy B(\rho) = 2\pi m/Q \neq 0$. We conclude that $B = \phi_B \delta(x)\delta(y)$, which would give an infinite contribution to the magnetic energy. Next we calculate the magnetic energy for a trial $A = m\rho/Q(\rho^2+\delta^2)$.

$$B(\rho) = \frac{m}{Q}\frac{2\delta^2}{(\rho^2+\delta^2)^2}, \qquad \frac{\epsilon_0 c^2}{2}\int 2\pi\rho d\rho B^2 = \frac{2\pi\epsilon_0 c^2 m^2\delta^4}{Q^2}\int_0^\infty \frac{du}{(u+\delta^2)^4} = \frac{2\pi\epsilon_0 c^2 m^2}{3Q^2\delta^2} \quad (3)$$

which diverges quadratically as $\delta \to 0$.

18. With simple trial functions for A, f that have the necessary behaviors at $\rho \to 0, \infty$, as determined in the previous problem, make a variational estimate of the energy per unit length T of a magnetic vortex. Discuss its dependence on λ, Q, f_0. (Some possible trials are $A = m\rho/(\rho^2+\alpha^2)Q$ and $f^2 = f_0^2\rho^2/(\rho^2+\beta^2)$, but feel free to invent your own trials.)

Solution Call the four terms in the energy per unit length T_1, T_2, T_3, T_4 respectively. Then

$$T_1 = \frac{2\pi\epsilon_0 c^2 m^2}{3Q^2\alpha^2}, \quad T_2 = \frac{\pi f_0^2}{2}, \quad T_3 = \frac{\pi f_0^2 m^2\alpha^4}{(\beta^2-\alpha^2)^2}\left[\ln\frac{\alpha^2}{\beta^2} + \frac{\beta^2-\alpha^2}{\alpha^2}\right], \quad T_4 = \frac{1}{4}\pi\lambda f_0^4\beta^2 \quad (4)$$

We notice that T_2 is independent of α, β and T_3 only depends on the ratio $\xi = \beta/\alpha$. Thus if we choose α, ξ as our variationsl parameters, we can minimize w.r.t. α by minimizing $T_1 + T_4$, which is minimized for $\alpha^4 = 8\epsilon_0 c^2 m^2/3Q^2\xi^2 f_0^4\lambda$. At this value of α

$$T_1 + T_4 = \frac{\pi\lambda f_0^4\xi^2}{2}\sqrt{\frac{8\epsilon_0 c^2 m^2}{3Q^2\xi^2 f_0^4\lambda}} = \pi f_0^2\zeta\xi|m|, \qquad \zeta = \sqrt{\frac{2\epsilon_0 c^2\lambda}{3Q^2}} \quad (5)$$

and we find

$$T = \pi f_0^2\left(\frac{1}{2} + \zeta\xi|m| + m^2\left[\frac{1}{\xi^2-1} - \frac{\ln\xi^2}{(\xi^2-1)^2}\right]\right) \quad (6)$$

This function can be easily analyzed graphically for different choices for ζ. We can go further for the limiting situations with ζ large and small. For $\zeta \gg |m|$ ($\lambda \gg Q^2/\epsilon_0 c^2$) the minimum occurs at small ξ. Explicitly $\xi \approx 2|m|/\zeta$. Then

$$T_{min} = \pi f_0^2\left(\frac{1}{2} + 2m^2 + m^2\left[-1 + 2\ln\frac{\zeta}{2|m|}\right]\right) = \pi f_0^2\left(\frac{1}{2} + m^2 + 2m^2\ln\frac{\zeta}{2|m|}\right) \quad (7)$$

For $\zeta \ll 2|m|$ the minimum occurs for large $\xi \approx (2|m|/\zeta)^{1/3}$.

$$T_{min} \sim \pi f_0^2\left(\frac{1}{2} + \frac{3}{2^{2/3}}\zeta^{2/3}|m|^{4/3}\right) \quad (8)$$

19. J, Problem 12.15. The Proca equation is just the Maxwell equation for the scalar electrodynamics we discussed in class, for the case of a constant scalar field $\phi = \phi_0$. $\epsilon_0 c\partial_\nu F^{\mu\nu} = -\mu^2 A^\mu$. The photon "mass" parameter is $\mu \equiv \sqrt{2Q^2|\phi_0|^2}$ in this problem. Please solve this problem using SI units throughout. This means that the quoted fields in a) and b) will contain appropriate factors of ϵ_0, μ_0, c, and 4π!

Solution

a)The static Green function for the Proca equation, $(-\nabla^2+\mu^2)\boldsymbol{A}=\mu_0\boldsymbol{J}$, is that of a point charge at $\boldsymbol{r}'$.

$$(-\nabla^2+\mu^2)G(\boldsymbol{r},\boldsymbol{r}')=\delta(\boldsymbol{r}-\boldsymbol{r}'),\qquad G=\frac{e^{-\mu|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \tag{9}$$

Then for the bound current density $\boldsymbol{J}=\nabla\times\boldsymbol{M}=-\boldsymbol{m}\times\nabla f$ we have the solution

$$\boldsymbol{A} = -\mu_0\boldsymbol{m}\times\int d^3r'\frac{\nabla' f(\boldsymbol{r}')e^{-\mu|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} = -\mu_0\boldsymbol{m}\times\nabla\int d^3r'\frac{f(\boldsymbol{r}')e^{-\mu|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \tag{10}$$

where an integration by parts was done after which the replacement $\nabla'\to-\nabla$ was valid because the derivatives are applied to a function of $\boldsymbol{r}-\boldsymbol{r}'$.

b) Putting $f=\delta(\boldsymbol{r}')$

$$\begin{aligned}\boldsymbol{B} &= \nabla\times\boldsymbol{A}=-\frac{\mu_0}{4\pi}(\boldsymbol{m}\nabla^2-(\boldsymbol{m}\cdot\nabla)\nabla)\frac{e^{-\mu|\boldsymbol{r}|}}{|\boldsymbol{r}|}\\ &= -\frac{\mu_0}{4\pi}\left(\mu^2\boldsymbol{m}\frac{e^{-\mu r}}{r}-\boldsymbol{m}\delta(\boldsymbol{r})+\boldsymbol{m}\cdot\nabla\frac{\boldsymbol{r}e^{-\mu r}}{r^3}(1+\mu r)\right)\end{aligned}$$

$$\begin{aligned}\boldsymbol{m}\cdot\nabla\frac{\boldsymbol{r}e^{-\mu r}}{r^3}(1+\mu r) &= \boldsymbol{m}\frac{e^{-\mu r}}{r^3}(1+\mu r)-\frac{\boldsymbol{r}\boldsymbol{m}\cdot\boldsymbol{r}}{r}\frac{e^{-\mu r}}{r^4}(3+3\mu r+\mu^2r^2)\\ \boldsymbol{B} &= -\frac{\mu_0}{4\pi}\left(\left(1+\mu r+\frac{1}{3}\mu^2r^2\right)(\boldsymbol{m}-3\hat{r}\hat{r}\cdot\boldsymbol{m})\frac{e^{-\mu r}}{r^3}+\frac{2}{3}\mu^2r^2\frac{e^{-\mu r}}{r^3}\right)\end{aligned}$$

where we dropped the $\delta(\boldsymbol{r})$ because $\boldsymbol{r}\neq 0$.

c)The quoted limit translates to $2\mu^2/3<4\times10^{-3}(1/r^2+\mu/r+\mu^2/3)$ or $(1-2\times10^{-3})\mu^2r^2<6\times10^{-3}(1+\mu r)$. This is roughly $\mu^{-1}>13r$ or 13 earth radii. Trnaslating to the photon mass, we have $m_\gamma=\mu\hbar/c<\hbar/13cR\approx4\times10^{-48}$g.

20. J, Problem 12.18.

Solution: We call the energy momentum tensor $T^{\mu\nu}$ instead of $\Theta^{\mu\nu}$. Just as in the current case, the argument is based on the result of integrating $\partial_\mu T^{\mu\nu}(x)=0$ over the spacetime region bounded by two hyperplanes: one at $x^0=T_1$ where x^μ are the space-time coordinates of observer 1, and the other at $x'^0=T_2'$ where x'^μ are the space-times coordinates of observer 2 moving at constant speed with respect to observer 1. We use Gauss' Law, assuming that the contributions of the boundary at spatial infinity vanish. Then

$$0 = \int d^4x\partial_\mu T^{\mu\nu}(x)=\int_{T_2'}d^3Sn_{2\mu}T^{\mu\nu}(x)-\int_{T_1}d^3xT^{0\nu}(\boldsymbol{x},T_1) \tag{11}$$

Here we have identified d^3x as d^3S for the boundary at $x^0=T_1$. Similarly d^3S for the boundary at $x'^0=T_2'$ is accordingly d^3x' where x'^μ are related to x^μ by a Lorentz transformation.

In order to express the corresponding integrand in terms of x' variables we use the tensor transformation laws: $T'^{\alpha\beta}(x') = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu T^{\mu\nu}(x)$. Also $n'_{2\alpha} = \eta^0{}_\alpha = (\Lambda^{-1})^\mu{}_\alpha n_{2\mu}$, so it follows that $T'^{0\beta}(x') = \Lambda^\beta{}_\nu n_{2\mu} T^{\mu\nu}(x)$. Then we have

$$\begin{aligned} p'^\beta(T'_2) &= \int d^3x' T'^{0\beta}(x') = \Lambda^\beta{}_\nu \int_{T'_2} d^3S n_{2\mu} T^{\mu\nu}(x) \\ &= \Lambda^\beta{}_\nu \int d^3x T^{0\nu}(x) = \Lambda^\beta{}_\nu p^\nu(T_1) \end{aligned}$$

Choosing $\Lambda = I$ we learn that p^ν is time independent. Then for general Λ the equation shows that the constants p^ν transform as the components of a 4-vector under Lorentz transformations.