Electromagnetic Theory II

Solution Set 6

Due: 24 February 2021

21. J, Problem 7.4.

Solution:

a) Take the interface to be the xy-plane and the incident wave to be proportional to $e^{ikz-i\omega t}$, with $k=\omega/c$:

$$\boldsymbol{E} = e^{-i\omega t}\begin{cases}(E_i e^{ikz}+E_r e^{-ikz})\hat{x} & z<0\\ E_t\hat{x}e^{ik'z} & z>0\end{cases} \tag{1}$$

$$\boldsymbol{B} = \frac{1}{i\omega}\nabla\times\boldsymbol{E} = e^{-i\omega t}\begin{cases}\frac{k}{\omega}(E_i e^{ikz}-E_r e^{-ikz})\hat{y} & z<0\\ \frac{k'}{\omega}E_t\hat{y}e^{ik'z} & z>0\end{cases} \tag{2}$$

The Ampere-Maxwell equation for the tranmitted wave gives

$$\frac{k'^2}{i\omega} = \mu_0\sigma - i\omega\mu_0\epsilon \tag{3}$$

$$k'^2 = \omega^2\mu_0\epsilon + i\mu_0\sigma\omega = k^2\frac{\epsilon}{\epsilon_0}+ik\sigma\sqrt{\frac{\mu_0}{\epsilon_0}} = k^2\epsilon_r+\frac{2i}{\delta^2} = \sqrt{k^4\epsilon_r^2+\frac{4}{\delta^4}}e^{i\tan^{-1}(2/k^2\delta^2\epsilon_r)}$$

$$k' = \left(k^4\epsilon_r^2+\frac{4}{\delta^4}\right)^{1/4}e^{i(1/2)\tan^{-1}(2/k^2\delta^2\epsilon_r)}$$

where $\epsilon_r=\epsilon/\epsilon_0$ and $\delta=\sqrt{2/\mu_0\omega\sigma}$. Continuity of $\boldsymbol{E}$ and $\boldsymbol{B}$ at $z=0$ gives

$$E_i+E_r = E_t, \qquad k(E_i-E_r)=k'E_t \tag{4}$$

$$r\equiv\frac{E_r}{E_i} = \frac{k-k'}{k+k'} = \frac{1-\sqrt{\epsilon_r+2i/(\delta k)^2}}{1+\sqrt{\epsilon_r+2i/(\delta k)^2}} \tag{5}$$

Clearly this amplitude is a complicated complex number in general. We could write the square root as $\alpha+i\beta$ and determine α,β by solving the pair of quadratic equations

$$\alpha^2-\beta^2 = \epsilon_r, \qquad \alpha\beta=\frac{1}{\delta^2k^2}$$

$$\alpha^4-\epsilon_r\alpha^2-\frac{1}{\delta^4k^4} = 0, \qquad \alpha^2=\frac{1}{2}\left(\epsilon_r+\sqrt{\epsilon_r^2+\frac{4}{\delta^4k^4}}\right)$$

$$\beta^2 = \frac{1}{2}\left(-\epsilon_r+\sqrt{\epsilon_r^2+\frac{4}{\delta^4k^4}}\right) \tag{6}$$

Then $|r|^2=((1-\alpha)^2+\beta^2)/((1+\alpha)^2+\beta^2)$, and the phase is $-\tan^{-1}(\beta/(1-\alpha))-\tan^{-1}(\beta/(1+\alpha))$.

b) For a poor conductor $\sigma \to 0$ so $k\delta >> 1$. Then

$$r \sim \frac{1-\sqrt{\epsilon_r}-i/(\delta^2 k^2 \epsilon_r)}{1+\sqrt{\epsilon_r}+i/(\delta^2 k^2 \epsilon_r)} \to \frac{1-\sqrt{\epsilon_r}}{1+\sqrt{\epsilon_r}} \tag{7}$$

On the other hand for a good conductor $k\delta << 1$ and

$$r \sim \frac{\delta k - \sqrt{2i}}{\delta k + \sqrt{2i}} = \frac{\delta k - 1 - i}{\delta k + 1 + i} \tag{8}$$

$$R = |r|^2 \sim \frac{(1-\delta k)^2+1}{(1+\delta k)^2+1} \approx 1 - 2\delta k = 1 - 2\delta\frac{\omega}{c} \tag{9}$$

22. J, Problem 7.13.

Solution

a) For a plasma the dielectric constant is $\epsilon = \epsilon_0(1-\omega_p^2/\omega^2)$ corresponding to an index of refraction $n = \sqrt{1-\omega_p^2/\omega^2} < 1$. The critical angle of incidence for the onset of total internal reflection, assuming the light comes from unit index of refraction solves $\sin i_c = n = \sqrt{1-\omega_p^2/\omega^2}$. Total internal reflection occurs for $i > i_c$.

b) Let L be the distance to the radio antenna. Then if the wave is reflected from the ionosphere at altitude h, the angle of incidence is $\sin i = L/2\sqrt{h^2+L^2/4}$. Apparently the critical angle in the case at hand is then $\sin i_0 = 5/\sqrt{34}$. Thus $\omega_p^2 = \omega^2(1-25/34) = 9\omega^2/34$. Here $\omega = 2\pi c/\lambda = 2\pi \times 10^8/7\text{s}^{-1}$. On the other hand (recall the Bohr radius $a_0 = \hbar/m_e c\alpha \approx 0.5 \times 10^{-10}$m)

$$\omega_p^2 = \frac{e^2 N_e}{m_e \epsilon_0} = \frac{4\pi\hbar c\alpha N_e}{m_e} = 4\pi\alpha^2 a_0 c^2 N_e$$

$$N_e = \frac{\omega_p^2}{4\pi\alpha^2 a_0 c^2} \approx \frac{9}{34}\left(\frac{2\pi}{7}\right)^2 \times 10^{16} \frac{(137)^2}{4\pi(0.5\times10^{-10})\cdot 9\times10^{16}}\text{m}^{-3} \approx 0.74\times10^{12}\text{m}^{-3}$$

which is between the night and day values given.

23. In our class discussion of plane waves in a plasma in the presence of a uniform magnetic field $\boldsymbol{B}_0$, we derived a formula for the displacement field $\boldsymbol{D}$ in terms of the electric field:

$$\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P} = \epsilon_0\left[\boldsymbol{E} - \frac{\omega_p^2}{\omega^2-\omega_B^2}\left(\boldsymbol{E} - i\boldsymbol{E}\times\frac{\boldsymbol{\omega}_B}{\omega} - \frac{\boldsymbol{\omega}_B}{\omega}\frac{\boldsymbol{\omega}_B}{\omega}\cdot\boldsymbol{E}\right)\right] \tag{10}$$

where $\boldsymbol{\omega}_B = e\boldsymbol{B}_0/m$ is the cyclotron frequency for the magnetic field. We can express this relation in terms of a matrix dielectric constant $D^k = \epsilon_{kl}E^l$. Find the three eigenvalues and their eigenvectors of the matrix ϵ_{kl}. Relate your results to the propagation of a plane wave in the plasma.

Solutions: Let e be an eigenvalue of ϵ_{kj}. Then

$$e\boldsymbol{E} \;=\; \epsilon_0\left[\boldsymbol{E}-\frac{\omega_p^2}{\omega^2-\omega_B^2}\left(\boldsymbol{E}-i\boldsymbol{E}\times\frac{\boldsymbol{\omega}_B}{\omega}-\frac{\boldsymbol{\omega}_B}{\omega}\frac{\boldsymbol{\omega}_B}{\omega}\cdot\boldsymbol{E}\right)\right] \tag{11}$$

Taking the scalar product of both sides with $\boldsymbol{\omega}_B$, we learn that

$$e\boldsymbol{E}\cdot\boldsymbol{\omega}_B \;=\; \boldsymbol{E}\cdot\boldsymbol{\omega}_B\epsilon_0\left[1-\frac{\omega_p^2}{\omega^2-\omega_B^2}\left(1-\frac{\boldsymbol{\omega}_B^2}{\omega^2}\right)\right]=\boldsymbol{E}\cdot\boldsymbol{\omega}_B\epsilon_0\left[1-\frac{\omega_p^2}{\omega^2}\right] \tag{12}$$

Thus if $\boldsymbol{E}\cdot\boldsymbol{\omega}_B\neq 0$, we deduce that $e\equiv e_0=\epsilon_0(1-\omega_p^2/\omega^2)$ the permittivity of a plasma in zero magnetic field. It is easily seen that the eigenvector belonging to this eigenvalue is simply $\boldsymbol{E}_0=\boldsymbol{\omega}_B/\omega_B$. On the other hand if $e\neq e_0$, it follows that $\boldsymbol{E}\cdot\boldsymbol{\omega}_B=0$. In that case, choose coordinate axes so that $\boldsymbol{\omega}_B=\omega_B\hat{z}$ and write out the eigenvalue equation in components

$$\begin{aligned}eE^x &= \epsilon_0\left[E^x-\frac{\omega_p^2}{\omega^2-\omega_B^2}\left(E^x-iE^y\frac{\omega_B}{\omega}\right)\right]\\ eE^y &= \epsilon_0\left[E^y-\frac{\omega_p^2}{\omega^2-\omega_B^2}\left(E^y+iE^x\frac{\omega_B}{\omega}\right)\right]\end{aligned}$$

with characteristic equation and solutions

$$0=\left(e-\epsilon_0\frac{\omega^2-\omega_B^2-\omega_p^2}{\omega^2-\omega_B^2}\right)^2-\frac{\epsilon_0\omega_p^4\omega_B^2}{\omega^2(\omega^2-\omega_B^2)^2}$$

$$e_\pm=\epsilon_0\frac{\omega^2-\omega_B^2-\omega_p^2}{\omega^2-\omega_B^2}\pm\epsilon_0\frac{\omega_p^2\omega_B}{\omega(\omega^2-\omega_B^2)}=\epsilon_0\left(1-\omega_p^2\frac{1\mp\omega_B/\omega}{\omega^2-\omega_B^2}\right)\;=\;\epsilon_0\left(1-\frac{\omega_p^2}{\omega(\omega\pm\omega_B)}\right)$$

which are just the two permittivities associated with a plane wave in a plasma propagating parallel to a magnetic field. The eigenvector belonging to $e_\pm$ satisfies

$$E_\pm^x\left(\frac{\omega_p^2}{\omega^2-\omega_B^2}-\frac{\omega_p^2}{\omega(\omega\pm\omega_B)}\right)\;=\;iE_\pm^y\frac{\omega_p^2\omega_B/\omega}{\omega^2-\omega_B^2} \tag{13}$$

which implies $E_\pm^y=\mp iE_\pm^x$. The corresponding eigenvector is $(\hat{x}\mp i\hat{y})/\sqrt{2}$, which for a plane wave propagating in the z direction are just the two states of circular polarization. In summary we list the three eigenvalues along with the eigenvector of each:

$$\begin{aligned}e_0 &= \epsilon_0\left(1-\frac{\omega_p^2}{\omega^2}\right), && \hat{z}\\ e_\pm &= \epsilon_0\left(1-\frac{\omega_p^2}{\omega(\omega\pm\omega_B)}\right), && \frac{\hat{x}\mp i\hat{y}}{\sqrt{z}}\end{aligned} \tag{14}$$

where $\boldsymbol{B} = B\hat{z}$.

24. To understand the behavior of a wave packet,

$$\psi(\boldsymbol{r}, t) = \int d^3k f(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega(\boldsymbol{k})t}, \tag{15}$$

it is useful to have an example where all the integrals can be done explicitly for all t. Consider a Gaussian form $f(\boldsymbol{k}) = Ae^{-(\boldsymbol{k}-\boldsymbol{k}_0)^2/(2\delta^2)}$. Assuming a quadratic behavior $\omega = \boldsymbol{k}^2/2\alpha$ for the frequency renders the integrand a Gaussian for all t.

a) Calculate the integral that defines ψ in this case, putting the answer in a form $K(t)e^{i\boldsymbol{k}_0\cdot\boldsymbol{r} - (1-it\delta^2/\alpha)(\boldsymbol{r}-\boldsymbol{r}_0(t))^2/\Delta^2(t)}$ and find explicit expressions for $K(t), \Delta(t), \boldsymbol{r}_0(t)$.

Solution We evaluate

$$\begin{aligned}
\psi &= \int d^3k e^{-(\boldsymbol{k}-\boldsymbol{k}_0)^2/2\delta^2 + i\boldsymbol{k}\cdot\boldsymbol{r} - ik^2t/2\alpha} = e^{i\boldsymbol{k}_0\cdot\boldsymbol{r} - ik_0^2t/2\alpha} \int d^3k e^{-\boldsymbol{k}^2(1/\delta^2 + it/\alpha)/2 + i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{k}_0t/\alpha)} \\
&= e^{i\boldsymbol{k}_0\cdot\boldsymbol{r} - ik_0^2t/2\alpha} \int d^3k \exp\left\{-\left(\boldsymbol{k} - \frac{i(\boldsymbol{r}-\boldsymbol{r}_0)}{(1/\delta^2 + it/\alpha)}\right)^2 (1/\delta^2 + it/\alpha)/2 - \frac{1}{2}\frac{(\boldsymbol{r}-\boldsymbol{r}_0(t))^2}{(1/\delta^2 + it/\alpha)}\right\} \\
&= \frac{(2\pi)^{3/2}}{(1/\delta^2 + it/\alpha)^{3/2}} e^{i\boldsymbol{k}_0\cdot\boldsymbol{r} - ik_0^2t/2\alpha} \exp\left\{-\frac{1}{2}\frac{(\boldsymbol{r}-\boldsymbol{r}_0(t))^2}{(1/\delta^2 + it/\alpha)}\right\} \\
&= \frac{(2\pi)^{3/2}}{(1/\delta^2 + it/\alpha)^{3/2}} e^{i\boldsymbol{k}_0\cdot\boldsymbol{r} - ik_0^2t/2\alpha} \exp\left\{-\frac{1}{2}\frac{(\boldsymbol{r}-\boldsymbol{r}_0(t))^2}{(1/\delta^2 + \delta^2t^2/\alpha^2)} + \frac{it\delta^2}{2\alpha}\frac{(\boldsymbol{r}-\boldsymbol{r}_0(t))^2}{(1/\delta^2 + \tilde{\delta}^2t^2/\alpha^2)}\right\}
\end{aligned}$$

Here $\boldsymbol{r}_0(t) = \boldsymbol{k}_0 t/\alpha$ and we can identify the spatial width of the packet as $\Delta(t) = \sqrt{2/\delta^2 + 2\delta^2t^2/\alpha^2}$. The factor $K(t)$ includes a complicated time dependent phase, but its magnitude is simply $|K| = (8\pi^2)^{3/4}\delta^{3/2}/\Delta^{3/2}(t)$.

b) Show that $d\boldsymbol{r}_0/dt = \boldsymbol{v}_g$ is just the group velocity we have introduced in describing the motion of wave packets with sharply peaked wave number distributions.

Solution: The group velocity is $d\omega/dk = \boldsymbol{k}/\alpha$. Evaluating this at the center of the packet gives $\boldsymbol{k}_0/\alpha = d\boldsymbol{r}_0/dt$.

c) Discuss how the wave function spreads with time. Find the time it takes to double in width. What are the restrictions on an initial wave packet for which there is negligible spreading throughout a scattering process in which there is a distance L from production location of the incoming beam to the location of the detector.?

Solution: The time to double in width is given by $\Delta(t) = 2\Delta(0)$ or $1/\delta^2 + \delta^2t^2/\alpha^2 = 4/\delta^2$ or $t = \sqrt{3}\alpha/\delta^2$. The time for the experiment to complete is $L/v_g \sim L\alpha/k_0$. Significant spreading takes a time $\sim \alpha/\delta^2$, so we would require $L/k_0 \ll 1/\delta^2 = \Delta^2(0)$. This means that $\lambda/L \ll \Delta(0)^2/L^2$. For probing atomic systems we would need λ smaller or comparable to the atomic size, so this is not a very stringent requirement!