Electromagnetic Theory II

Solution Set 7

Due: 10 March 2021

25. J, Problem 7.15.

Solution

a) For a plasma $\omega = \sqrt{k^2c^2 + \omega_p^2}$ so the group velocity is $d\omega/dk = ck/\omega \sim c(1 - \omega_p^2/2\omega^2)$ for $\omega \gg \omega_p$. Then we calculate the arrival time from

$$ct = \int_0^R \frac{cdz}{v_g} \approx \int_0^R dz(1 + \omega_p^2/2\omega^2) = R + \frac{1}{2\omega^2}\int_0^R \omega_p^2 = R + \frac{e^2}{2\epsilon_0 m_e \omega^2}\int_0^R N_e(z). \quad (1)$$

b) Since in the presence of a magnetic field photons of different helicity see different index of refraction the relative phase of the two helicities in a linearly polarized state will change with z:

$$(\epsilon_1 + i\epsilon_2)e^{i\omega n_- z/c} + (\epsilon_1 - i\epsilon_2)e^{i\omega n_+ z/c} = 2e^{i\omega(n_- + n_+)z/2c}\left(\epsilon_1 \cos\frac{\omega(n_- - n_+)z}{2c} - \epsilon_2 \sin\frac{\omega(n_- - n_+)z}{2c}\right)$$

Thus $(n_- - n_+)/2 \approx -e^3 N_e B_\parallel / 2\epsilon_0 m_e^2 \omega^3$, so that

$$\delta\theta = \int_0^R dz \frac{\omega(n_- - n_+)}{2c} \approx -\frac{e^3}{2\epsilon_0 c m_e^2 \omega^2}\int dz N_e(z) B_\parallel(z) \quad (2)$$

c) The frequency dependence of $t(\omega)/R$ and $\delta\theta(\omega)/R$ can determine $\langle N_e \rangle$ and $\langle N_e B_\parallel \rangle$. The lower frequency components of each pulse will arrive later. So each pulse will be smeared in arrival time with frequency shifting toward the red. Plotting arrival time versus ω can confirm the $1/\omega^2$ dependence and its coefficient will give $\langle N_e \rangle R$. Similarly the amount of Faraday rotation will vary with frequency and arrival time. This will yield a measurement of $\langle N_e B_\parallel \rangle / \langle N_e \rangle$ which is a kind of weighted average of $B_\parallel$. We need to assume linear polarization initially.

26. J, Problem 7.22.

Solutions:

a)For this part we have Im $\epsilon = \lambda\epsilon_0$ for $\omega_1 < \omega < \omega_2$ and $= 0$ otherwise. Thus we need to do

$$\int_{\omega_1}^{\omega_2} d\omega' \frac{\lambda\omega'}{\omega'^2 - \omega^2} = \frac{\lambda}{2}\ln\frac{\omega_2^2 - \omega^2}{\omega_1^2 - \omega^2} = \frac{\lambda}{2}\ln\left|\frac{\omega_2^2 - \omega^2}{\omega_1^2 - \omega^2}\right| \quad (3)$$

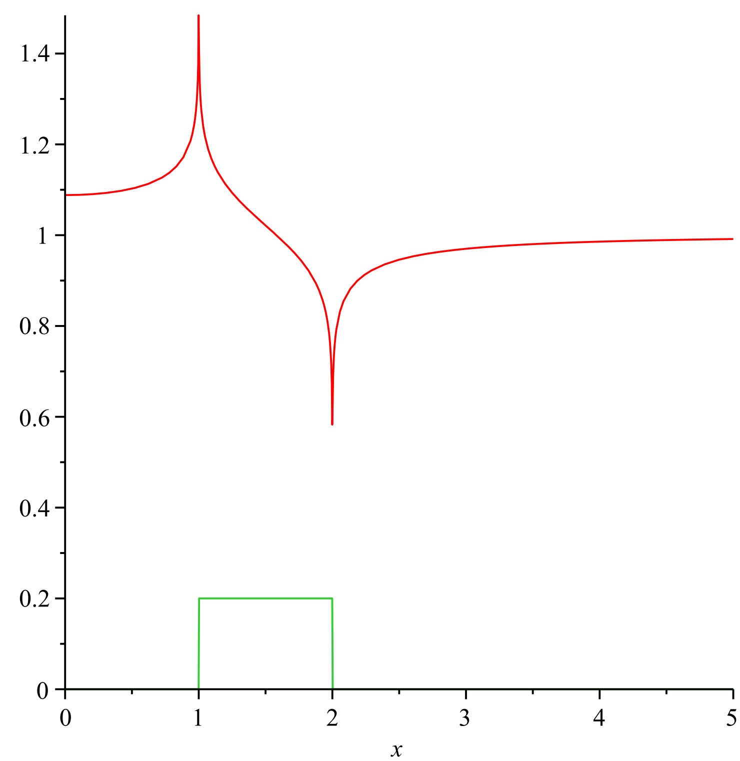
which is obviously valid for $\omega < \omega_1$ of $\omega > \omega_2$, because there is no singularity in the integrand and the principal part prescription is not necessary. But for $\omega_1 < \omega < \omega_2$ we have to use the principal part prescription:

$$\begin{aligned}
P\int_{\omega_1}^{\omega_2} d\omega' \frac{\lambda\omega'}{\omega'^2-\omega^2} &= \left\{\int_{\omega_1}^{\omega-\delta} + \int_{\omega+\delta}^{\omega_2}\right\} d\omega' \frac{\lambda\omega'}{\omega'^2-\omega^2} \\
&= \frac{\lambda}{2}\left[\ln\frac{(\omega-\delta)^2-\omega^2}{\omega_1^2-\omega^2} + \ln\frac{\omega_2^2-\omega^2}{(\omega+\delta)^2-\omega^2}\right] \\
&\to \frac{\lambda}{2}\left[\ln\frac{2\delta\omega}{\omega^2-\omega_1^2} + \ln\frac{\omega_2^2-\omega^2}{2\omega\delta}\right] \\
&= \frac{\lambda}{2}\ln\frac{\omega_2^2-\omega^2}{\omega^2-\omega_1^2} = \frac{\lambda}{2}\ln\left|\frac{\omega_2^2-\omega^2}{\omega_1^2-\omega^2}\right| \qquad (4)
\end{aligned}$$

Thus in all cases we can write

$$\mathrm{Re}\,\frac{\epsilon}{\epsilon_0} = 1+\frac{\lambda}{\pi}\ln\left|\frac{\omega_2^2-\omega^2}{\omega_1^2-\omega^2}\right|, \qquad \mathrm{Im}\,\frac{\epsilon}{\epsilon_0} = \lambda[\theta(\omega-\omega_1)-\theta(\omega-\omega_2)] \qquad (5)$$

The imaginary part is simply a square shape $\mathrm{Im}\epsilon = \lambda\epsilon_0$ for $\omega_1 < \omega < \omega_2$ and $= 0$ otherwise. The real part increases from $1 + (\lambda/\pi)\ln(\omega_2^2/\omega_1^2)$ to $+\infty$ as ω goes from 0 to ω_1. It then decreases from $+\infty$ to $-\infty$ as ω goes from ω_1 to ω_2. Finally it increases from $-\infty$ to 1 as ω goes from ω_2 to ∞. The singular behavior at ω_1 and ω_2 is caused by the sharp discontinuities in $\mathrm{Im}\epsilon$.



b) This time the needed integral, which we write in the form of J (7.19), is

$$P\int_{-\infty}^{\infty} d\omega' \frac{\lambda\gamma\omega'}{(\omega'-\omega)[(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2]}$$
$$= \int_{-\infty}^{\infty} d\omega' \left(\frac{1}{\omega'-\omega-i\delta} - \pi i\delta(\omega'-\omega)\right)\frac{\lambda\gamma\omega'}{[(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2]}$$
$$= \int_{-\infty}^{\infty} d\omega' \frac{1}{\omega'-\omega-i\delta}\frac{\lambda\gamma\omega'}{[(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2]} - \frac{\pi i\lambda\gamma\omega}{[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]} \tag{6}$$

as $\delta \to 0+$. Now write

$$\frac{\lambda\gamma\omega'}{[(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2]} = \frac{i\lambda/2}{\omega_0^2-\omega'^2+i\gamma\omega'} - \frac{i\lambda/2}{\omega_0^2-\omega'^2-i\gamma\omega'} \tag{7}$$

The first term on the right is analytic in the lower half ω' plane, where as the second term is analytic in the lower half plane. Thus we close the contour integral of the first term in the lower half plane, and close the contour integral of the second term in the upper half plane. The added semi-circle contours contribute nothing at infinite radius. The factor multiplying these terms is a pure pole at $\omega' = \omega + i\delta$ in the upper half plane. Thus the integral of the first term gives zero, whereas that of the second term gives the residue of this pole.

$$P\int_{-\infty}^{\infty} d\omega' \frac{\lambda\gamma\omega'}{(\omega'-\omega)[(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2]}$$
$$= \frac{\pi\lambda}{\omega_0^2-(\omega+i\delta)^2-i\gamma(\omega+i\delta)} - \frac{\pi i\lambda\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$
$$\to \frac{\pi\lambda}{\omega_0^2-\omega^2-i\gamma\omega} - \frac{\pi i\lambda\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} = \frac{\pi\lambda(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} \tag{8}$$

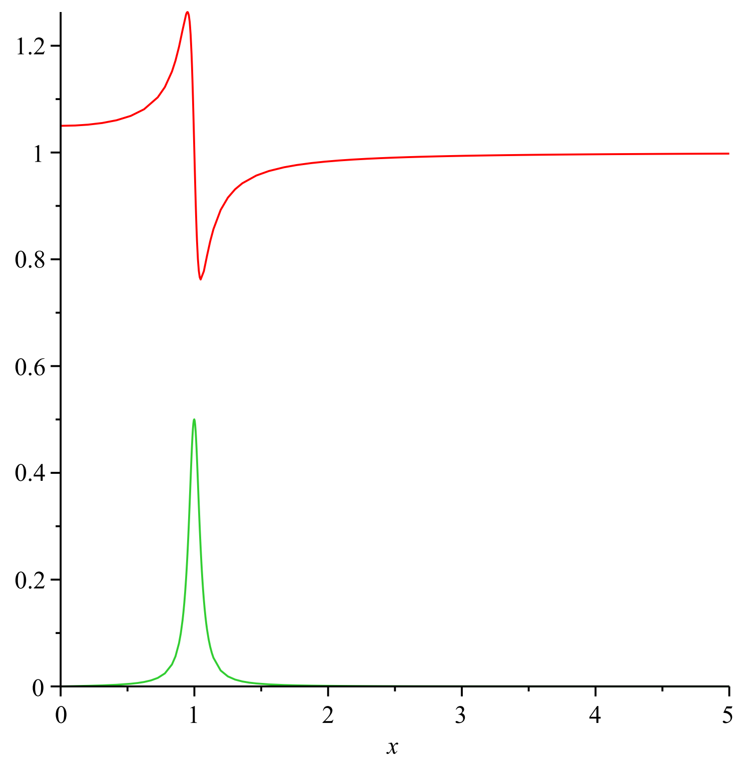
Thus

$$\mathrm{Re}\frac{\epsilon}{\epsilon_0} = 1 + \frac{\lambda(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} \tag{9}$$

Note that in this case

$$\frac{\epsilon}{\epsilon_0} = \mathrm{Re}\frac{\epsilon}{\epsilon_0} + i\mathrm{Im}\frac{\epsilon}{\epsilon_0} = 1 + \frac{\lambda(\omega_0^2-\omega^2+i\gamma\omega)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} = 1 + \frac{\lambda}{\omega_0^2-\omega^2-i\gamma\omega} \tag{10}$$

reproducing a single resonance version of our oscillator model of a dielectric. Aplot would look like the first resonance part of J, Figure 7.8.


1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
x

27. J, Problem 7.27.

Solution:

a) First rewrite

$$\begin{aligned}\boldsymbol{E}\times\boldsymbol{B} &= \boldsymbol{E}\times(\nabla\times\boldsymbol{A}) = E^i\nabla A^i - E^i\nabla^i\boldsymbol{A} = E^i\nabla A^i - \nabla^i(E^i\boldsymbol{A}) + \boldsymbol{A}\nabla\cdot\boldsymbol{E} \\ &= E^i\nabla A^i - \nabla^i(E^i\boldsymbol{A}) \end{aligned} \tag{11}$$

using Gauss' law $\nabla\cdot\boldsymbol{E}=0$. Then we find

$$\begin{aligned}\boldsymbol{L} &= \frac{1}{\mu_0 c^2}\int d^3x\boldsymbol{r}\times(E^i\nabla A^i - \nabla^i(E^i\boldsymbol{A})) \\ &= \frac{1}{\mu_0 c^2}\int d^3x[E^i(\boldsymbol{r}\times\nabla)A^i - \nabla^i(E^i\boldsymbol{r}\times\boldsymbol{A}) - \boldsymbol{A}\times(\boldsymbol{E}\cdot\nabla)\boldsymbol{r}] \\ &= \frac{1}{\mu_0 c^2}\int d^3x[E^i(\boldsymbol{r}\times\nabla)A^i - \boldsymbol{A}\times\boldsymbol{E}] - \oint dS\boldsymbol{n}\cdot\boldsymbol{E}\boldsymbol{r}\times\boldsymbol{A}\end{aligned} \tag{12}$$

where we used Gauss theorem to write the middle term as a surface integral, which tends to zero as the surface is taken to infinity:

$$\boldsymbol{L} = \frac{1}{\mu_0 c^2}\int d^3x[E^i(\boldsymbol{r}\times\nabla)A^i + \boldsymbol{E}\times\boldsymbol{A}] \tag{13}$$

b) From $\boldsymbol{E} = -\nabla\phi - \dot{\boldsymbol{A}}$, we derive the expansions

$$\begin{aligned}\boldsymbol{E} &= \sum_\lambda\int\frac{d^3k}{(2\pi)^3}\omega\left[i\boldsymbol{\epsilon}_\lambda a_\lambda(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t} + c.c.\right] \\ \boldsymbol{E}\times\boldsymbol{A} &= \sum_{\lambda\lambda'}\int\frac{d^3k}{(2\pi)^3}\omega(\boldsymbol{k})\frac{d^3k'}{(2\pi)^3}\left[i\boldsymbol{\epsilon}_\lambda a_\lambda(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t} + c.c.\right]\times\left[\boldsymbol{\epsilon}'_\lambda a'_\lambda(\boldsymbol{k}')e^{i\boldsymbol{k}'\cdot\boldsymbol{r}-i\omega' t} + c.c.\right]\end{aligned}$$

The integral of this expression over $\boldsymbol{r}$ produces delta functions as follows

$$\begin{aligned}\int d^3x e^{i\boldsymbol{k}'\cdot\boldsymbol{r}-i\omega' t}e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t} &= (2\pi)^3\delta(\boldsymbol{k}'+\boldsymbol{k})e^{-2i\omega t} \\ \int d^3x e^{i\boldsymbol{k}'\cdot\boldsymbol{r}-i\omega' t}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}+i\omega t} &= (2\pi)^3\delta(\boldsymbol{k}'-\boldsymbol{k})\end{aligned}$$

and their complex conjugates. The right side of the first equation is zero after time averaging, and ditto for the complex conjugate. Thus only the integral on the second line and its complex conjugate survive time averaging. Thus we find

$$\boldsymbol{L}_{\text{Spin}} = \frac{1}{\mu_0 c^2}\sum_{\lambda\lambda'}\int\frac{d^3k}{(2\pi)^3}\omega(\boldsymbol{k})\left[i\boldsymbol{\epsilon}_\lambda a_\lambda(\boldsymbol{k})\times\boldsymbol{\epsilon}^*_{\lambda'}a^*_{\lambda'}(\boldsymbol{k}) + c.c.\right] \tag{14}$$

Finally consider the cross product $\boldsymbol{\epsilon}_\lambda(\boldsymbol{k}) \times \boldsymbol{\epsilon}^*_{\lambda'}(\boldsymbol{k})$. Because of its definition $\boldsymbol{\epsilon}^*_\pm = \boldsymbol{\epsilon}_\mp$. Thus the cross product is zero unless $\lambda' = \lambda$. In that case we have

$$i\omega\boldsymbol{\epsilon}_\pm \times \boldsymbol{\epsilon}^*_\pm = \frac{i\omega}{2}(\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2) \times (\boldsymbol{\epsilon}_1 \mp i\boldsymbol{\epsilon}_2) = \mp i^2\omega\boldsymbol{\epsilon}_1 \times \boldsymbol{\epsilon}_2 = \pm kc\hat{k} = \pm c\boldsymbol{k} \quad (15)$$

Then we have

$$\begin{aligned}\boldsymbol{L}_{\text{Spin}} &= \frac{1}{\mu_0 c^2}\int \frac{d^3k}{(2\pi)^3}\omega(\boldsymbol{k})\sum_\lambda [i\boldsymbol{\epsilon}_\lambda \times \boldsymbol{\epsilon}^*_\lambda a^*_\lambda a_\lambda + c.c.] = \frac{1}{\mu_0 c^2}\int \frac{d^3k}{(2\pi)^3}\boldsymbol{k}c\left[|a_+|^2 - |a_-|^2 + c.c.\right] \\ &= \frac{2}{\mu_0 c}\int \frac{d^3k}{(2\pi)^3}\boldsymbol{k}[|a_+|^2 - |a_-|^2]\end{aligned}$$

The energy density is

$$\frac{\epsilon_0}{2}(\dot{\boldsymbol{A}})^2 + \frac{1}{2\mu_0}(\nabla \times \boldsymbol{A})^2 = \frac{1}{2\mu_0 c^2}(\dot{\boldsymbol{A}})^2 + \frac{1}{2\mu_0}(\nabla A_i)^2 - \frac{1}{2\mu_0}\nabla_i A_j(\nabla_j A_i) \quad (16)$$

The last term contributes zero to the energy integral in Coulomb gauge. Using the plane wave expansions and doing the integral over space, we get the total energy

$$\begin{aligned}U &= \frac{1}{2\mu_0 c^2}\sum_{\lambda\lambda'}\int \frac{d^3k}{(2\pi)^3}k^2c^2[\boldsymbol{\epsilon}_\lambda \cdot \boldsymbol{\epsilon}^*_{\lambda'} a_\lambda a^*_{\lambda'} + c.c.] + \frac{1}{2\mu_0}\sum_{\lambda\lambda'}\int \frac{d^3k}{(2\pi)^3}\boldsymbol{k}^2[\boldsymbol{\epsilon}_\lambda \cdot \boldsymbol{\epsilon}^*_{\lambda'} a_\lambda a^*_{\lambda'} + c.c.] \\ &= \frac{2}{\mu_0 c}\int \frac{d^3k}{(2\pi)^3}\omega k\sum_\lambda |a_\lambda|^2 = \frac{2}{\mu_0 c}\int \frac{d^3k}{(2\pi)^3}\omega k[|a_+|^2 + |a_-|^2]\end{aligned}$$

Comparing the expressions for U and $\boldsymbol{L}_{\text{Spin}}$, we see that the contribution of mode $\boldsymbol{k}$ to the angular momentum is just $\pm\hat{k}/\omega = \pm\hbar\hat{k}/(\hbar\omega)$ times the contribution of that mode to the energy. Since $\hbar\omega$ is the energy of a photon, we conclude that $\pm\hbar$ is its spin along the direction of motion. This component of spin is called the helicity.

28. J, Problem 8.2.

Solution:

a) For TEM modes $\nabla_\perp \cdot \boldsymbol{E}_\perp = 0$ and $\nabla_\perp \times \boldsymbol{E}_\perp = 0$. The second eq means we can write $\boldsymbol{E}_\perp = -\nabla_\perp\phi$, whence the first gives $\nabla^2_\perp\phi = 0$. $\boldsymbol{E}_\parallel = 0$ on the boundaries implies ϕ =constant on the boundaries. The unique solution of this electrostatics problem is $\phi = C_1 \ln\rho + C_2$, $\boldsymbol{E}_\perp = -C_1\hat{\rho}/\rho$, and $C_1 = -aE_0$, where aE_0 is the electric field at the inner boundary $\rho = a$. Then

$$\begin{aligned}\boldsymbol{H}_\perp &= \frac{1}{\mu}\boldsymbol{B}_\perp = \frac{k}{\mu\omega}\hat{z} \times \boldsymbol{E}_\perp = \sqrt{\frac{\epsilon}{\mu}}\frac{aE_0}{\rho}\hat{\varphi} \\ \langle\boldsymbol{S}\rangle &= \frac{1}{2}\text{Re } \boldsymbol{E} \times \boldsymbol{H}^* = \hat{z}\sqrt{\frac{\epsilon}{\mu}}\frac{a^2E_0^2}{\rho^2}, \\ P &= \int_a^b \rho d\rho \int_0^{2\pi} d\varphi\hat{z} \cdot \langle\boldsymbol{S}\rangle = \pi a^2E_0^2\sqrt{\frac{\epsilon}{\mu}}\ln\frac{b}{a} = \pi a^2H_0^2\sqrt{\frac{\mu}{\epsilon}}\ln\frac{b}{a}\end{aligned}$$

b) Applying the formula for the loss per unit length we have

$$
\begin{aligned}
-\frac{dP}{dz} &= \frac{1}{2\sigma\delta}\left(\oint_{\rho=a} dl + \oint_{\rho=b} dl\right)\frac{a^2H_0^2}{\rho^2} = \frac{\pi a^2 H_0^2}{\sigma\delta}\left[\frac{1}{a}+\frac{1}{b}\right] \\
\frac{1}{P}\frac{dP}{dz} &= -\sqrt{\frac{\epsilon}{\mu}}\frac{1}{\sigma\delta\ln(b/a)}\left[\frac{1}{a}+\frac{1}{b}\right], \qquad P(z) = P_0 e^{-2\beta z} \\
\beta &= \sqrt{\frac{\epsilon}{\mu}}\frac{1}{2\sigma\delta\ln(b/a)}\left[\frac{1}{a}+\frac{1}{b}\right] \qquad (17)
\end{aligned}
$$

c) From Ampere's Law $I_a = 2\pi\rho H = 2\pi a H_0$ where I_a is the current carried by the inner conductor. By definition

$$
V_a - V_b = \int_a^b d\rho E(\rho) = E_0 a \ln\frac{b}{a} \qquad (18)
$$

So we find $Z = V/I_a = (1/2\pi)\sqrt{\mu/\epsilon}\ln(b/a)$.

d) From c) we saw that the current carried by the inner conductor is $I_a = 2\pi a H_0$, and it also follows from Ampere's law that the current carried by the outer conductor is $I_b = -I_a$. Since these are amplitudes of oscillating currents the time averages of the squared current is $\langle I_a^2\rangle = (1/2)\Re I_a I_a^* = 2\pi^2 a^2 H_0^2$. On the other hand the power loss per unit length is $\pi a H_0^2/\sigma\delta$ at $\rho = a$ but $\pi a^2 H_0^2/b\sigma\delta$ at $\rho = b$. Thus the resistance per unit length at a is $R_a = \text{Loss}/I_a^2 = 1/2\pi a\sigma\delta$ while $R_b = 1/2\pi b\sigma\delta$. The series resistance per unit length is then $R_a + R_b$, the desired result.

The inductance can be inferred from the magnetic energy $U_B = (1/2)LI^2 = \frac{1}{2}\int d^3x\mu H^2$. The inductance per unit length is then

$$
\frac{dL}{dz} = \frac{1}{\langle I\rangle^2}\int d^2x\mu H^2 = \frac{2\pi}{2\pi^2 a^2 H_0^2}\int \rho d\rho\mu H^2 \qquad (19)
$$

Now $H = H_0 a/\rho$ for $a < \rho < b$, but is exponentially damped as it penetrates the conductor: $H \approx H_0 e^{-(a-\rho)/\delta}$ for $\rho < a$ and $H \approx (aH_0/b)e^{-(\rho-b)/\delta}$ for $\rho > b$. Thus

$$
\begin{aligned}
\frac{dL}{dz} &\approx \frac{1}{2\pi a^2}\left[\mu\int d\rho\frac{a^2}{\rho} + \mu_c\int_b^\infty \frac{a^2 d\rho}{b}e^{-2(\rho-b)/\delta} + \mu_c\int_{-\infty}^a a d\rho e^{-2(a-\rho)/\delta}\right] \\
&\approx \frac{1}{2\pi}\left[\mu\ln\frac{b}{a} + \mu_c\frac{\delta}{2b} + \mu_c\frac{\delta}{2a}\right]
\end{aligned}
$$