Electromagnetic Theory II

Solution Set 8

Due: 17 March 2021

29. J, Problem 8.4. Circular cylindrical wave guide with finite σ

Solution:

a) For a right circular cylinder, the Helmholtz equation satisfied by E^z, B^z separates in polar coordinates ρ, φ. On solutions of the form $f_m(\rho)e^{im\varphi}$

$$\nabla_\perp^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} \to \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{m^2}{\rho^2} \tag{1}$$

and the radial equation is solved by Bessel functions $f_m = J_m(\rho\sqrt{\omega^2\epsilon\mu - k^2})$. The boundary conditions at $\rho = R$ are Dirichlet for TM and Neumann for TE modes. Let x_{mn} be the zeroes of $J_m(x)$ and y_{mn} be the zeroes of $J'_m(x)$. Then the allowed frequencies for TM and TE modes are

$$\omega_{mn}^{\rm TM} = \frac{1}{\sqrt{\epsilon\mu}}\sqrt{k^2 + \frac{x_{mn}^2}{R^2}}, \qquad \omega_{mn}^{\rm TE} = \frac{1}{\sqrt{\epsilon\mu}}\sqrt{k^2 + \frac{y_{mn}^2}{R^2}} \tag{2}$$

The first few x, y's are:

$$\begin{aligned} x_{0n} &= 2.405, 5.520, \cdots; \qquad x_{1n} = 3.832, 7.016, \cdots; \qquad x_{2n} = 5.136, \cdots \\ y_{0n} &= 3.832, 7.016, \cdots; \qquad y_{1n} = 1.841, 5.331, \cdots; \qquad y_{2n} = 3.045, 6.706, \cdots \\ y_{11} &< x_{01} < y_{21} < x_{11} = y_{01} < x_{21} \end{aligned} \tag{3}$$

The dominant cutoff frequency is the TE mode $y_{11}/R\sqrt{\epsilon\mu} \approx 1.841c/R$ and the ratios of the next 4 modes to the dominant one are

$$\frac{x_{01}}{y_{11}} \approx 1.306, \quad \frac{y_{21}}{y_{11}} \approx 1.659, \quad \frac{x_{11}}{y_{11}} = \frac{y_{01}}{y_{11}} \approx 2.081, \quad \frac{x_{21}}{y_{11}} \approx 2.790 \tag{4}$$

b) To evaluate attenuation coefficients we need some integrals

$$\begin{aligned} \int_0^R \rho d\rho J_m^2(\rho x_{mn}/R) &= \frac{R^2}{x_{mn}^2}\int_0^{x_{mn}} x dx J_m^2(x) = \frac{R^2}{2}J_m'^2(x_{mn}), \qquad \text{TM} \\ \int_0^R \rho d\rho J_m^2(\rho y_{mn}/R) &= \frac{R^2}{y_{mn}^2}\int_0^{y_{mn}} x dx J_m^2(x) = \frac{R^2}{2}\left(1 - \frac{m^2}{y_{mn}^2}\right)J_m^2(y_{mn}), \quad \text{TE} \end{aligned}$$

For the TM case we need

$$\oint dl \frac{R^2}{x_{mn}^2}|\partial_n J_m|^2 = 2\pi R J_m'^2(x_{mn}) = \frac{4\pi}{R}\int_0^R \rho d\rho J_m^2(\rho x_{mn}/R) = \frac{2}{R}\int d^2x |J_m|^2$$

so that

$$\begin{aligned}\beta_{\rm TM} &= -\frac{1}{2P}\frac{dP}{dz} = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{-1/2}\frac{2}{R}\\ &= \frac{1}{\sigma\delta_{mn}}\sqrt{\frac{\epsilon}{\mu}}\sqrt{\frac{\omega}{\omega_{mn}}}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{-1/2}\frac{1}{R}\end{aligned}$$

For the TE case the numerator involves instead

$$\oint dl\left[\frac{R^2}{y_{mn}^2}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)|\nabla_\parallel J_m|^2+\frac{\omega_{mn}^2}{\omega^2}J_m^2\right] = 2\pi R\left[\frac{m^2}{y_{mn}^2}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)+\frac{\omega_{mn}^2}{\omega^2}\right]J_m^2(y_{mn})$$

$$= \frac{2}{R}\left[\frac{m^2}{y_{mn}^2}+\frac{\omega_{mn}^2}{\omega^2}\left(1-\frac{m^2}{y_{mn}^2}\right)\right]\left(1-\frac{m^2}{y_{mn}^2}\right)^{-1}\int d^2x J_m^2$$

$$\beta_{\rm TE} = \frac{1}{\sigma\delta_{mn}}\sqrt{\frac{\epsilon}{\mu}}\sqrt{\frac{\omega}{\omega_{mn}}}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{-1/2}\frac{1}{R}\left[\frac{m^2}{y_{mn}^2-m^2}+\frac{\omega_{mn}^2}{\omega^2}\right]$$

30. **Spherical cavity** In this problem we investigate some of the normal modes for electromagnetic fields with harmonic time dependence (*i.e.* $\propto e^{-i\omega t}$) in a *spherical* cavity of radius a in a perfect conductor. Use polar coordinates with origin at the center of the cavity. When $k \equiv \omega/c \neq 0$, the source-free Maxwell equations can be reduced in two equivalent ways to:

$$\begin{aligned} &\text{I}: &\boldsymbol{B} = \frac{1}{ikc}\nabla\times\boldsymbol{E} \qquad &\nabla\cdot\boldsymbol{E}=0 \qquad (-\nabla^2-k^2)\boldsymbol{E}=0\\ \text{or}\quad &\text{II}: &\boldsymbol{E} = -\frac{c}{ik}\nabla\times\boldsymbol{B} \qquad &\nabla\cdot\boldsymbol{B}=0 \qquad (-\nabla^2-k^2)\boldsymbol{B}=0\end{aligned}$$

a) For $\omega \neq 0$ the fields must be strictly zero in the bulk of the conductor. Why? Use Maxwell's equations to prove that the fields in the cavity must then satisfy the boundary conditions $\boldsymbol{E}_t = \boldsymbol{B}_n = 0$ at $r = a$.

Solution: A perfect conductor instantly neutralizes any electric field which implies also that any time varying magnetic field ($\omega \neq 0$) is also cancelled. The sourceless Maxwell equations then imply the desired boundary conditions as explained in the lecture notes.

b) Consider the following forms for TE ($\hat{\mathbf{r}}\cdot\boldsymbol{E}=0$) and TM ($\hat{\mathbf{r}}\cdot\boldsymbol{B}=0$) modes:

$$\text{TE}: \quad \boldsymbol{E}_{\rm TE} = f(r)\hat{\mathbf{r}}\times\boldsymbol{C}e^{-i\omega t}; \qquad \text{TM}: \quad \boldsymbol{B}_{\rm TM} = g(r)\hat{\mathbf{r}}\times\boldsymbol{D}e^{-i\omega t},$$

where $\boldsymbol{C}$ and $\boldsymbol{D}$ are constant vectors. It is clearly convenient to use form (I), (II) of Maxwell's equations for the TE, TM cases respectively. Show that $\boldsymbol{E}_{TE}$, $\boldsymbol{B}_{TM}$ automatically have zero divergence and that all boundary conditions will be satisfied in the TE case if $f(a) = 0$ and in the TM case if $(rg(r))'|_{r=a} = 0$.

Solution:

$$\nabla \cdot \boldsymbol{E}_{TE} = \boldsymbol{C} \cdot (\nabla \times \frac{f(r)\boldsymbol{r}}{r}) = 0 \tag{5}$$

because the curl of any central force field is zero. Ditto for $\boldsymbol{B}_{TM}$. The parallel component of $\boldsymbol{E}_{TE}$ is singled out by $\hat{r} \times \boldsymbol{E}_{TE} \propto f(a)(\hat{r} \cdot \boldsymbol{C}\hat{r} - \boldsymbol{C})$ which will vanish only if $f(a) = 0$. Then $\hat{r} \cdot \boldsymbol{B}_{TE} \propto f\hat{r} \cdot \boldsymbol{C}$ also vanishes at $r = a$. For the TM case, it is immediate that $\hat{r} \cdot \boldsymbol{B}_{TM} = 0$. Then we compute

$$\begin{aligned} \boldsymbol{E}_{TM} &\propto \nabla \times (\frac{g\boldsymbol{r}}{r} \times \boldsymbol{D}) = \boldsymbol{D} \cdot \nabla \frac{g\boldsymbol{r}}{r} - \boldsymbol{D}\nabla \cdot \frac{g\boldsymbol{r}}{r} = \boldsymbol{r}\boldsymbol{D} \cdot \hat{r} \left[\frac{g}{r}\right]' - \boldsymbol{D}\left((2\frac{g}{r} + r\left[\frac{g}{r}\right]'\right) \\ \boldsymbol{r} \times \boldsymbol{E}_{TM} &\propto \boldsymbol{r} \times \boldsymbol{D}\frac{g + rg'}{r} = \boldsymbol{r} \times \boldsymbol{D}\frac{(rg)'}{r} = 0 \end{aligned} \tag{6}$$

if $(rg)' = 0$ at $r = a$.

c) In the TE case show that last equation in (I) will be satisfied provided $f(r)$ satisfies the differential equation

$$\left(-\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{2}{r^2} - k^2\right) f = 0,$$

where $k = \omega/c$. Obviously the TM case works the same way using form (II) if g satisfies the same equation. *Hint*: Recall that the components of $\hat{\mathbf{r}}$ can be expressed as linear combinations of the spherical harmonics $Y_{1,\pm1}(\theta,\phi), Y_{1,0}(\theta,\phi)$, and $-\nabla^2 Y_{lm}(\theta,\phi) = (l(l+1)/r^2)Y_{lm}(\theta,\phi)$.

Solution: Since the angular dependence in $\hat{r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, is carried by a linear combiination of Y_{1m}, the Laplacian separates into a radial and angular part, with $-\nabla^2 Y_{lm} = (l(l+1)/r^2)Y_{lm} \to (2/r^2)Y_{1m}$ for $l = 1$. Then

$$\begin{aligned} \nabla^2(f(r)\hat{r} \times \boldsymbol{C}) &= \hat{r} \times \boldsymbol{C}\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{2}{r^2}\right) f(r) \\ -k^2 f(r) &= \left(\frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{2}{r^2}\right) f(r) \end{aligned} \tag{7}$$

one can easily check that

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}r = \frac{1}{r}\frac{\partial}{\partial r}\left(1 + r\frac{\partial}{\partial r}\right) = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} \tag{8}$$

Obviously $g(r)$ satisfies the same equation.

d) Show that the spherical Bessel function

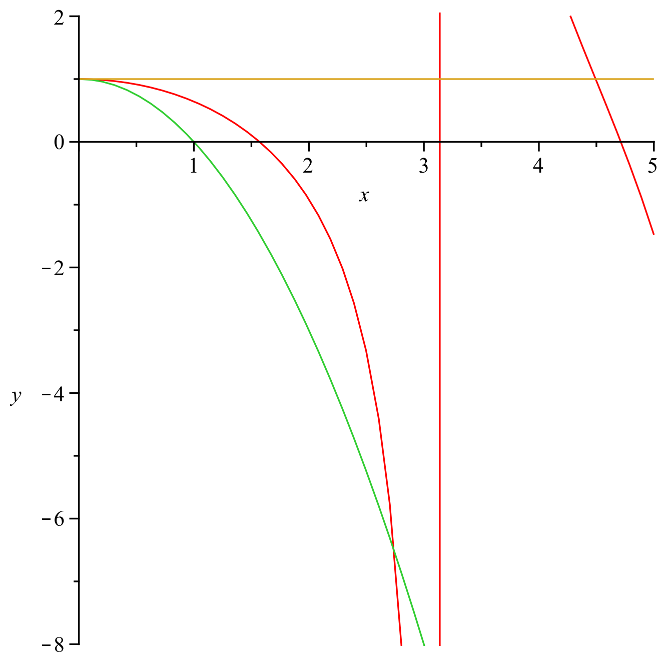
$$j_1(kr) \equiv \frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}$$

is the unique solution of this differential equation that is regular at $r = 0$. Imposing the appropriate boundary condition, obtain a graphical solution for the lowest frequency in each case. Which frequency is lower, TE or TM?

Solution: We calculate

$$\begin{aligned}
\frac{1}{r}\frac{\partial^2}{\partial r^2} r j_1 &= \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(\frac{\sin kr}{k^2 r} - \frac{\cos kr}{k}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\cos kr}{kr} - \frac{\sin kr}{k^2 r^2} + \sin kr\right) \\
&= -\frac{\sin kr}{r^2} - \frac{\cos kr}{kr^3} - \frac{\cos kr}{kr^3} + 2\frac{\sin kr}{k^2 r^4} + k\cos kr = \left(-k^2 + \frac{2}{r^2}\right) j_1
\end{aligned}$$

as desired. There are 2 linearly independent solutions with different small r behavior: the other one blows up like r^{-2} at small r. (For general l the two behaviors are r^l, r^{-l+1}. Graphical solution. For TE case $j_1(ka) = 0$ implies $ka \cot ka = 1$. For TM case $(r j_1(kr))' = 0$ implies $ka \cot ka = 1 - (ka)^2$. So on a plot of $x \cot x$ we superpose $1 - x^2$ and 1:



We see the lowest TM solution is $ka \approx 2.75$ and the lowest TE solution is $ka \approx 4.5$. Thus the lowest mode is TM.

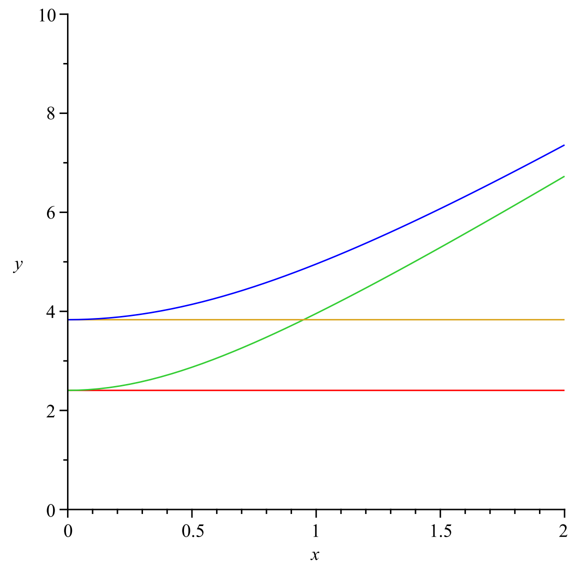
31. J, Problem 8.6 Circular cylindrical cavity.

Solution:

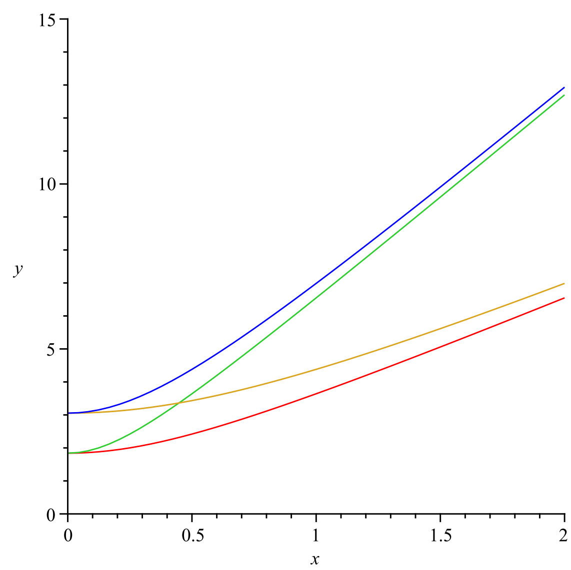
a)For TM modes $E_z = E_0 J_m(\rho\gamma_{mn}) \cos(p\pi z/L)$, with $\gamma_{mn} R = x_{mn}$ a zero of J_m and $p = 0, 1, 2, \ldots$, whereas for TE modes $H_z = H_0 J_m(\rho\gamma_{mn}) \sin(p\pi z/L)$ with $\gamma_{mn} R = y_{mn}$ a zero of J'_m and $p = 1, 2, \ldots$. Then

$$\begin{aligned}
\sqrt{\epsilon\mu} R\omega^{\rm TM} &= \sqrt{x_{mn}^2 + \frac{p^2\pi^2 R^2}{L^2}}, \qquad p = 0, 1, 2, \cdots \\
\sqrt{\epsilon\mu} R\omega^{\rm TE} &= \sqrt{y_{mn}^2 + \frac{p^2\pi^2 R^2}{L^2}}, \qquad p = 1, 2, \cdots
\end{aligned}$$

The first four TM mode frequencies versus $x = R/L$:



The first four TE mode frequencies versus $x = R/L$:



b)For $R/L = 2/3$ the lowest TE frequency is $\sqrt{\epsilon\mu}R\omega^{\rm TE} = \sqrt{4\pi^2/9 + (1.841)^2} \approx 2.788$ which is higher than the lowest TM mode $\sqrt{\epsilon\mu}R\omega^{\rm TM} = 2.405$ (for $p = 0$). The formula for Q for a $p = 0$ TM mode is

$$Q = \frac{\mu L}{\mu_c\delta(1 + \xi CL/2A)} \to \frac{\mu L}{\mu_c\delta(1 + L/R)} \tag{9}$$

where we used $\xi = 1$ for the right circular cylinder as found in problem 8.4. For the dimensions of this cavity $\omega = 2.405c/R = 3.5 \times 10^{10}{\rm s}^{-1}$. For copper $\sigma^{-1} = 1.68 \times 10^{-8}\Omega$m and $\delta = 8.6 \times 10^{-5}$cm so assuming $\mu = \mu_c = \mu_0$, we find $Q \approx 6/(5 \cdot 8.6) \times 10^5 \approx 1.4 \times 10^4$.

32. J, Problem 8.9 Variational method for cavity modes.

Solution:

a) We vary $k^2[E]$ w.r.t. δE:

$$\delta k^2 = \frac{\int d^3x[\delta\boldsymbol{E}^* \cdot (\nabla \times (\nabla \times \boldsymbol{E}) - k^2\boldsymbol{E}) + \boldsymbol{E}^* \cdot (\nabla \times (\nabla \times \delta\boldsymbol{E} - k^2\delta\boldsymbol{E}))]}{\int d^3x|\boldsymbol{E}|^2}$$

We see immediately that the coefficient of $\delta\boldsymbol{E}^*$ is zero if and only if $\nabla\times(\nabla\times\boldsymbol{E}) = k^2\boldsymbol{E}$. After two int by parts the coefficient of $\delta\boldsymbol{E}$ will be zero if and only if $\nabla \times (\nabla \times \boldsymbol{E}^*) = k^2\boldsymbol{E}^*$. The surface terms on the int by parts will vanish provided both $\boldsymbol{E}, \delta\boldsymbol{E}$ satisfy perfect conductor boundary conditions.

b) We evaluate k^2 for $\boldsymbol{E} = \hat{z}E_0\cos(\pi\rho/2R)$, $\nabla \times \boldsymbol{E} = \hat{\varphi}(\pi E_0/2R)\sin(\pi\rho/2R)$

$$k^2 = \frac{\pi^2}{4R^2}\frac{\int d^3x \sin^2(\pi\rho/2R)}{\int d^3x\cos^2(\pi\rho/2R)} = \frac{\pi^2}{4R^2}\frac{\int_0^R \rho d\rho\sin^2(\pi\rho/2R)}{\int_0^R \rho d\rho\cos^2(\pi\rho/2R)} = \frac{\pi^2}{4R^2}\frac{\pi^2+4}{\pi^2-4} \approx 2.415 \tag{10}$$

We see that the estimate is only slightly above the exact 2.405....

c) Next we try $\boldsymbol{E} = \hat{z}E_0[1+\alpha(\rho/R)^2-(1+\alpha)(\rho/R)^4$, which satisfies the boundary conditions. Then

$$\begin{aligned} k^2(\alpha) &= \frac{20}{R^2}\frac{\alpha^2+4\alpha+6}{\alpha^2+7\alpha+16} = \frac{20}{R^2}\left[1 - \frac{3\alpha+10}{\alpha^2+7\alpha+16}\right] \\ \frac{dk^2}{d\alpha} &= -\frac{20}{R^2}\left[\frac{-3\alpha^2-20\alpha-22}{(\alpha^2+7\alpha+16)^2}\right] = 0 \\ \alpha + \frac{-20 \pm \sqrt{400 - 4\cdot 22}}{6} &= \frac{-10 \pm \sqrt{34}}{3}, \qquad k^2 = \frac{80}{R^2}\frac{17 \mp 2\sqrt{34}}{68 \pm \sqrt{34}} \end{aligned}$$

The smaller value of k^2 is the upper sign, and we find

$$kR = \sqrt{80\frac{17 - 2\sqrt{34}}{68 + \sqrt{34}}} \approx 2.405 \tag{11}$$

even closer to the exact value!