Electromagnetic Theory II

Solution Set 9

Due: 25 March 2021

33. J, Problem 8.18. Orthogonality of soutions of the Helmholtz equation.

Solution:

a) We prove that solutions of the Helmholtz equation $-\nabla_\perp^2\psi = \gamma^2\psi$, with different γ are orthogonal using Green's theorem

$$\int d^2x[\psi\nabla_\perp^2\phi - \phi\nabla_\perp^2\psi] = \oint dl\boldsymbol{n}\cdot(\psi\nabla_\perp\phi - \phi\nabla_\perp\psi) \tag{1}$$

If both ϕ, ψ satisfy Dirichlet conditions (TM) or both satisfy Neumann conditions (TE), the right side is 0. It then follows that

$$(\gamma_\lambda^2 - \gamma_\mu^2)\int d^2x\psi_\lambda\psi_\mu = 0 \tag{2}$$

So orthogonality follows when $\gamma_\lambda^2 \neq \gamma_\mu^2$. If $\gamma_\lambda^2 = \gamma_\mu^2$ for $\mu \neq \lambda$, one can always take linear combinations that ensure orthogonality (Gramm-Schmidt procedure).

b)For TE case

$$\begin{aligned}
\int d^2x\boldsymbol{E}_{\perp\lambda}\cdot\boldsymbol{E}_{\perp\mu} &= -\frac{\omega^2}{\gamma_\lambda^2\gamma_\mu^2}\int d^2x(\hat{z}\times\nabla_\perp)B_{z\lambda}\cdot(\hat{z}\times\nabla_\perp)B_{z\mu} \\
&= -\frac{\omega^2}{\gamma_\lambda^2\gamma_\mu^2}\int d^2x\nabla_\perp B_{z\lambda}\cdot\nabla_\perp B_{z\mu} = -\frac{\omega^2}{\gamma_\lambda^2\gamma_\mu^2}\int d^2xB_{z\lambda}(-\nabla^2)_\perp B_{z\mu} \\
&= -\frac{\omega^2}{\gamma_\lambda^2}\int d^2xB_{z\lambda}B_{z\mu} = \delta_{\lambda\mu}
\end{aligned} \tag{3}$$

requires $\int d^2xH_{z\lambda}H_{z\mu} = -(\gamma_\lambda^2/\mu^2\omega^2)\delta_{\lambda\mu} = -(\gamma_\lambda^2/k_\lambda^2Z_\lambda^2)\delta_{\lambda\mu}$. which is 8.134(b). For the TM case

$$\int d^2x\boldsymbol{E}_{\perp\lambda}\cdot\boldsymbol{E}_{\perp\mu} = -\frac{k_\lambda^2}{\gamma_\lambda^2\gamma_\mu^2}\int d^2x\nabla_\perp E_{z\lambda}\cdot\nabla_\perp E_{z\mu} = -\frac{k_\lambda^2}{\gamma_\lambda^2}\int d^2xE_{z\lambda}E_{z\mu} = \delta_{\lambda\mu}$$

requires $\int d^2xE_{z\lambda}E_{z\mu} = (-\gamma_\lambda^2/k_\lambda^2)\delta_{\lambda\mu}$, which is 8.134a). If λ is TE and μ is TM

$$\begin{aligned}
\int d^2x\boldsymbol{E}_{\perp\lambda}\cdot\boldsymbol{E}_{\perp\mu} &= \frac{\omega k_\lambda}{\gamma_\lambda^2\gamma_\mu^2}\int d^2x(\hat{z}\times\nabla_\perp)B_{z\lambda}\cdot\nabla_\perp E_{z\mu} = -\frac{\omega k_\lambda}{\gamma_\lambda^2\gamma_\mu^2}\int d^2xE_z\nabla_\perp\cdot(\hat{z}\times\nabla_\perp)B_{z\lambda} \\
&= \frac{\omega k_\lambda}{\gamma_\lambda^2\gamma_\mu^2}\int d^2xE_z\hat{z}\cdot(\nabla_\perp\times\nabla_\perp)B_{z\lambda} = 0
\end{aligned} \tag{4}$$

Thus 8.131 and 8.134 are proved with consistent normalization. Finally, we have simply

$$\begin{aligned}
\int d^2x \boldsymbol{H}_{\perp\lambda}\cdot\boldsymbol{H}_{\perp\mu} &= \frac{1}{Z_\lambda Z_\mu}\int d^2x(\hat{z}\times\boldsymbol{E}_{\perp\lambda})\cdot(\hat{z}\times\boldsymbol{E}_{\perp\mu}) = \frac{1}{Z_\lambda Z_\mu}\int d^2x\boldsymbol{E}_{\perp\lambda}\cdot\boldsymbol{E}_{\perp\mu} = \frac{1}{Z_\lambda^2}\delta_{\lambda\mu} \\
\int d^2x\hat{z}\cdot(\boldsymbol{E}_{\perp\lambda}\times\boldsymbol{H}_{\perp\mu}) &= \frac{1}{Z_\mu}\int d^2x\hat{z}\cdot(\boldsymbol{E}_{\perp\lambda}\times(\hat{z}\times\boldsymbol{E}_{\perp\mu})) = \frac{1}{Z_\mu}\int d^2x\boldsymbol{E}_{\perp\lambda}\cdot\boldsymbol{E}_{\perp\mu} = \frac{1}{Z_\mu}\delta_{\lambda\mu}
\end{aligned}$$

which completes the derivations.

34. Two charges, $+q$ and $-q$, separated by a fixed distance d rotate at constant angular velocity ω in the xy-plane about the z-axis which bisects the line joining the charges.

a)Calculate the power radiated per unit solid angle by this system in the dipole approximation.

Solution: The electric dipole moment is

$$\boldsymbol{p} = q(\boldsymbol{r}_+(t)-\boldsymbol{r}_-(t)) = qd(\hat{x}\cos\omega t+\hat{y}\sin\omega t) = \text{Re }qd(\hat{x}+i\hat{y})e^{-i\omega t}$$

The complex vector $\boldsymbol{p} = qd(\hat{x}+i\hat{y})$ is the proper one to use in the electric dipole approximation

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{Z_0c^2k^4}{32\pi^2}|\hat{r}\times(\hat{r}\times\boldsymbol{p})|^2 = \frac{Z_0c^2k^4}{32\pi^2}(\boldsymbol{p}\cdot\boldsymbol{p}^* - |\hat{r}\cdot\boldsymbol{p}|^2) \\
&= \frac{Z_0c^2k^4}{32\pi^2}(2q^2d^2 - q^2d^2\sin^2\theta) = \frac{q^2d^2Z_0c^2k^4}{32\pi^2}(1+\cos^2\theta) \qquad (5)
\end{aligned}$$

where the observation direction is $\hat{r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ in spherical coordinates.

b)Discuss the polarization and relative intensity of the radiation emitted (i) in the plane of motion and (ii) along the axis of rotation.

Solution: The polarization is proportional to the complex electric field, which is in the complex direction $\hat{r}\times(\hat{r}\times\boldsymbol{p}) = \hat{r}\hat{r}\cdot\boldsymbol{p}-\boldsymbol{p}$. For observation in the xy plane we may take $\hat{r}=\hat{x}$ in which case the direction vector is $qd(\hat{x}-(\hat{x}+i\hat{y})) = -iqd\hat{y}$. That is linear polarization in the y direction. For observation down the axis of rotation $\hat{r}=\hat{z}$ so the polarization direction is $-qd(\hat{x}+i\hat{y})$, meaning circular polarizarion. Since the angular distribution is $1+\cos^2\theta$ The relative intensities observed in the xy-plane $(\theta=\pi/2)$ and down the axis $(\theta=0)$ is 1 to 2.

35. J, Problem 9.2

Solution: let a corner of the square with $+q$ follow the circular trajectory

$$\boldsymbol{r}(t) = (a/\sqrt{2})(\cos\omega t, \sin\omega t, 0)$$

Then the other $+q$ will be at $-\boldsymbol{r}(t)$. The negative charges $-q$ will then be at $\boldsymbol{r}(t+\pi/2\omega)$ and at $-\boldsymbol{r}(t+\pi/2\omega)$. The quadrupole moment tensor is then

$$Q_{jk} = \sum_i q_i(3r_i^j r_i^k - \delta_{jk}\boldsymbol{r}_i^2) = 3\sum_i q_i r_i^j r_i^k \tag{6}$$

since $\sum_i q_i = 0$. The only nonzero components are

$$Q_{11} = \frac{3a^2q}{2}(2\cos^2\omega t - 2\sin^2\omega t) = 3a^2q\cos 2\omega t = \text{Re }3a^2qe^{-2i\omega t}$$
$$Q_{22} = -3a^2q\cos 2\omega t = \text{Re }-3a^2qe^{-2i\omega t}, \qquad Q_{12} = Q_{21} = 3a^2q\sin 2\omega t = \text{Re }i3a^2qe^{-2i\omega t} \tag{7}$$

The complex quantities inside the real parts are the ones to use in the quadrupole radiation formula. We also note that the frequency is 2ω, so the wave number of radiation will be $k = 2\omega/c$. We first form the vector $\boldsymbol{Q}$ with components

$$\begin{aligned} Q^1 &= Q_{1k}\hat{r}^k = \sin\theta(Q_{11}\cos\varphi + Q_{12}\sin\varphi) = 3a^2q\sin\theta e^{i\varphi} \\ Q^2 &= Q_{2k}\hat{r}^k = \sin\theta(Q_{21}\cos\varphi + Q_{22}\sin\varphi) = 3ia^2q\sin\theta e^{i\varphi}, \qquad Q^3 = 0 \end{aligned} \tag{8}$$

Thus $\boldsymbol{Q} = 3a^2q\sin\theta e^{i\varphi}(1, i, 0)$. Then

$$|\hat{r}\times(\hat{r}\times\boldsymbol{Q})|^2 = \boldsymbol{Q}\cdot\boldsymbol{Q}^* - |\hat{r}\cdot\boldsymbol{Q}|^2 = 9a^4q^2\sin^2\theta(2-\sin^2\theta) = 9a^4q^2(1-\cos^4\theta) \tag{9}$$

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{9Z_0c^2k^6q^2a^4}{1152\pi^2}(1-\cos^4\theta) = \frac{Z_0c^2k^6q^2a^4}{128\pi^2}(1-\cos^4\theta) \\ P &= \frac{Z_0c^2k^6}{1440\pi}[|Q_{11}|^2 + |Q_{22}|^2 + 2|Q_{12}|^2] = \frac{Z_0c^2k^6q^2a^4}{40\pi} = \frac{8Z_0\omega^6q^2a^4}{5\pi c^4} \end{aligned} \tag{10}$$