Electromagnetic Theory II

Solution Set 10

Due: 31 March 2021

36. J, Problem 9.5.

Solution:

a) Starting with the exact scalar potential, we expand in $\boldsymbol{r'}$ to first order

$$\begin{aligned}\phi &= \frac{1}{4\pi\epsilon_0}\int d^3x'\rho(\boldsymbol{r'})\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r'}|}}{|\boldsymbol{r}-\boldsymbol{r'}|} \approx \frac{1}{4\pi\epsilon_0}\int d^3x'\rho(\boldsymbol{r'})\frac{e^{ik(r-\hat{r}\cdot\boldsymbol{r'})}}{(r-\hat{r}\cdot\boldsymbol{r'})}\\ &\approx \frac{e^{ikr}}{4\pi\epsilon_0 r}\int d^3x'\rho(\boldsymbol{r'})\left(1-ik\hat{r}\cdot\boldsymbol{r'}+\hat{r}\cdot\frac{\boldsymbol{r'}}{r}\right) = \frac{e^{ikr}}{4\pi\epsilon_0 r}\left(Q-\frac{\hat{r}}{r}\cdot\boldsymbol{p}(ikr-1)\right)\end{aligned}$$

The Q term is the monopole potential while the second term inside the large parentheses is gives the desired dipole scalar potential. As already discussed in our lecture notes and in the text of Jackson the electric dipole contribution to the vector potential just comes from the first term in the expansion of $e^{-ik\hat{r}\cdot\boldsymbol{r'}} \approx 1$ and the the leading contribution of $1/|\boldsymbol{r}-\boldsymbol{r'}| \approx 1/r$:

$$\boldsymbol{A} \approx \frac{\mu_0 e^{ikr}}{4\pi r}\int d^3x'\boldsymbol{J} = -\frac{\mu_0 e^{ikr}}{4\pi r}\int d^3x'\boldsymbol{r'}\nabla'\cdot\boldsymbol{J} = -\frac{\mu_0 e^{ikr}}{4\pi r}\int d^3x'\boldsymbol{r'}i\omega\rho = -i\omega\frac{\mu_0 e^{ikr}}{4\pi r}\boldsymbol{p}$$

b) The easiest field to calculate is $\boldsymbol{H}$:

$$\boldsymbol{H} = \frac{1}{\mu_0}\nabla\times\boldsymbol{A} = -i\omega\hat{r}\times\boldsymbol{p}\frac{d}{dr}\frac{e^{ikr}}{4\pi r} = k\omega\hat{r}\times\boldsymbol{p}\frac{e^{ikr}}{4\pi r}\left(1-\frac{1}{ikr}\right) \tag{1}$$

which agrees with the first of (9.18). The electric field is considerably more involved:

$$\begin{aligned}\boldsymbol{E} &= i\omega\boldsymbol{A}-\nabla\phi = \omega^2\frac{\mu_0 e^{ikr}}{4\pi r}\boldsymbol{p}+\nabla\left(\frac{e^{ikr}}{4\pi\epsilon_0}\left(\frac{ik}{r^2}-\frac{1}{r^3}\right)\boldsymbol{r}\cdot\boldsymbol{p}\right)\\ &= k^2\frac{e^{ikr}}{4\pi\epsilon_0 r}\boldsymbol{p}+\hat{r}\left(\frac{e^{ikr}}{4\pi\epsilon_0}\left(\frac{-k^2}{r^2}-\frac{ik}{r^3}-2\frac{ik}{r^3}+\frac{3}{r^4}\right)\boldsymbol{r}\cdot\boldsymbol{p}\right)+\left(\frac{e^{ikr}}{4\pi\epsilon_0}\left(\frac{ik}{r^2}-\frac{1}{r^3}\right)\boldsymbol{p}\right)\\ &= k^2\frac{e^{ikr}}{4\pi\epsilon_0 r}(\hat{r}\times\boldsymbol{p})\times\hat{r}+\frac{e^{ikr}}{4\pi\epsilon_0}\left(-\frac{ik}{r^2}+\frac{1}{r^3}\right)3\hat{r}\hat{r}\cdot\boldsymbol{p}+\frac{e^{ikr}}{4\pi\epsilon_0}\left(\frac{ik}{r^2}-\frac{1}{r^3}\right)\boldsymbol{p}\\ &= k^2\frac{e^{ikr}}{4\pi\epsilon_0 r}(\hat{r}\times\boldsymbol{p})\times\hat{r}+\frac{e^{ikr}}{4\pi\epsilon_0}\left(-\frac{ik}{r^2}+\frac{1}{r^3}\right)(3\hat{r}\hat{r}\cdot\boldsymbol{p}-\boldsymbol{p})\end{aligned}$$

which agrees with the second of (9.18).

37. J, Problem 9.8. Recall from Chapter 12 the identification of the current of angular momentum with $r^iT^{j\mu} - r^jT^{i\mu}$ where $T^{\rho\mu}$ is the symmetric energy momentum tensor. The point of Jackson's hint is that the angular momentum density $\epsilon \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B})$ as well as its flux has an extra factor of order r, and so with radiation zone fields, the flux has nominal large r behavior of $1/r$. Of course that leading behavior must cancel leaving a contribution of order $1/r^2$, which involves $1/r^2$ terms in the fields.

Solution:

a) The time averaged angular momentum density is

$$\mathrm{Re}\, \frac{1}{2c^2}\boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{H}^*) \quad = \quad \mathrm{Re}\, \frac{1}{2c^2}(\boldsymbol{r} \cdot \boldsymbol{H}^* \boldsymbol{E} - \boldsymbol{r} \cdot \boldsymbol{E} \boldsymbol{H}^*)$$

From J, (9.18), $\boldsymbol{r} \cdot \boldsymbol{H}^* = 0$ and

$$\boldsymbol{r} \cdot \boldsymbol{E} \quad = \quad \frac{\hat{r} \cdot \boldsymbol{p}}{4\pi\epsilon_0}\left(\frac{1}{r^2} - \frac{ik}{r}\right) e^{ikr}$$

The $1/r$ term in $\boldsymbol{E}$ drops out because it is perpendicular to $\hat{r} = \boldsymbol{n}$. Thus in the radiation zone the angular momentum density falls off as $1/r^2$ not $1/r$. Keeping only the leading terms as $r \to \infty$ we find

$$\mathrm{Re}\, \frac{1}{2c^2}\boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{H}^*) \quad \sim \quad \mathrm{Re}\, \frac{ik^3}{16\pi^2\epsilon_0 c r^2}\hat{r} \cdot \boldsymbol{p}\hat{r} \times \boldsymbol{p}^* = -\frac{k^3}{16\pi^2\epsilon_0 c r^2}\mathrm{Im}\ \hat{r} \cdot \boldsymbol{p}\hat{r} \times \boldsymbol{p}^*$$

Integrating the right side over angles and multiplying by r^2 gives the total angular momentum in a spherical shell of radius r and of thickness dr. Making use of $\int d\Omega \hat{r}^i\hat{r}^j = 4\pi/3$ leads to

$$\frac{d\boldsymbol{L}}{dr} \quad = \quad -\frac{k^3}{12\pi\epsilon_0 c}\mathrm{Im}\ \boldsymbol{p} \times \boldsymbol{p}^* = \frac{k^3}{12\pi\epsilon_0 c}\mathrm{Im}\ \boldsymbol{p}^* \times \boldsymbol{p}$$

Since this shell is expanding at the velocity of light, we get the time rate by multiplying by c

$$\frac{d\boldsymbol{L}}{dt} \quad = \quad c\frac{d\boldsymbol{L}}{dr} = \frac{k^3}{12\pi\epsilon_0}\mathrm{Im}\ \boldsymbol{p}^* \times \boldsymbol{p}$$

b) Dividing by the radiated power $P = c^2 Z_0 k^4 |\boldsymbol{p}|^2$ gives

$$\frac{1}{P}\frac{d\boldsymbol{L}}{dt} \quad = \quad \frac{1}{k\epsilon_0 c^2 Z_0}\frac{\mathrm{Im}\ \boldsymbol{p}^* \times \boldsymbol{p}}{|\boldsymbol{p}|^2} = \frac{1}{\omega}\frac{\mathrm{Im}\ \boldsymbol{p}^* \times \boldsymbol{p}}{|\boldsymbol{p}|^2}$$

The left side has the dimensions of action over energy. For a photon $E = \hbar\omega$ and its spin angular momentum is $\hbar$ so if a photon were emitted, this ratio would be $1/\omega$ exactly.

c) A charge revolving in a circle of radius a and frequency ω has the electric dipole moment

$$\begin{aligned}
\boldsymbol{p} &= \mathrm{Re}[qa(1,i,0)e^{-i\omega t}] \\
\boldsymbol{p}^* \times \boldsymbol{p} &= q^2a^2(\hat{x} - i\hat{y}) \times (\hat{x} + i\hat{y}) = 2iq^2a^2\hat{z} \\
\frac{dL_z}{dt} &= \frac{k^3 q^2 a^2}{6\pi\epsilon_0}, \qquad \frac{dL_x}{dt} = \frac{dL_y}{dt} = 0.
\end{aligned}$$

A charge oscillating on the z-axis has dipole moment $\boldsymbol{p} = \mathrm{Re}\ qa\hat{z}e^{-i\omega t}$, so $\boldsymbol{p}^* \times \boldsymbol{p} = 0$ so that no angular momentum will be radiated.

d) As shown in J, (9.35), (9.36), we get the fields of a magnetic dipole from those of an electric dipole by substituting $\boldsymbol{E} \to Z_0\boldsymbol{H}$, $Z_0\boldsymbol{H} \to -\boldsymbol{E}$, $\boldsymbol{p} \to \boldsymbol{m}/c$. the $-$ in the second substitution means that $\boldsymbol{E} \times \boldsymbol{H}^* \to -\boldsymbol{H} \times \boldsymbol{E}^* = \boldsymbol{E}^* \times \boldsymbol{H}$ which has the same real part as $\boldsymbol{E} \times \boldsymbol{H}^*$. Thus the only change in our results is the substitution $\boldsymbol{p} \to \boldsymbol{m}/c$.

$$\frac{d\boldsymbol{L}}{dt} = \frac{k^3}{12\pi\epsilon_0 c^2}\mathrm{Im}\ \boldsymbol{m}^* \times \boldsymbol{m}, \qquad \frac{1}{P}\frac{d\boldsymbol{L}}{dt} = \frac{1}{\omega}\frac{\mathrm{Im}\ \boldsymbol{m}^* \times \boldsymbol{m}}{|\boldsymbol{m}|^2}$$

38. J, Problem 9.11.

Solution: The total charge, electric dipole moment and magnetic dipole moment are all zero. The lowest moment is therefore the electric quadrupole moment tensor, which is diagonal with nonzero elements given by

$$\begin{aligned}
Q_{33} &= -2q(3z^2 - r^2) = -4qz^2 = -4qa^2\cos^2\omega t = -2qa^2(1 + \cos 2\omega t) \\
&= -2qa^2 - \mathrm{Re}\ 2qa^2e^{-2i\omega t} \\
Q_{11} = Q_{22} &= 2qr^2 = 2qz^2 = 2qa^2\cos^2\omega t = qa^2 + \mathrm{Re}\ qa^2e^{-2i\omega t}
\end{aligned}$$

Since we are assuming $ka \ll 1$ the radiation will be dominated by electric quadrupole radiation at frequency 2ω, so the wave number for the radiation is $k = 2\omega/c$, and its quadrupole amplitudes are $Q_{33} = -2Q_{11} = -2Q_{22} = -2qa^2$. The formula for the angular distribution of radiated power and total radiated power are

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{c^2k^6Z_0}{1152\pi^2}(Q_{ij}\hat{r}^jQ^*_{ip}\hat{r}^p - |Q_{ij}\hat{r}^i\hat{r}^J|^2) \\
&= \frac{(2\omega)^6Z_0}{1152c^4\pi^2}(Q_{11}^2\sin^2\theta + Q_{33}^2\cos^2\theta - |Q_{11}\sin^2\theta + Q_{33}\cos^2\theta|^2) \\
&= \frac{2\omega^6Z_0}{9c^4\pi^2}Q_{11}^2(1 + 3\cos^2\theta - |1 - 3\cos^2\theta\cos^2\theta|^2) = Q_{11}^2\frac{2\omega^6Z_0}{c^4\pi^2}\sin^2\theta\cos^2\theta \\
&= \frac{2q^2a^4\omega^6Z_0}{c^4\pi^2}\sin^2\theta\cos^2\theta \\
P &= \int d\omega\frac{dP}{d\Omega} = \frac{2q^2a^4\omega^6Z_0}{c^4\pi^2}2\pi\int_{-1}^{1}d\cos\theta(\cos^2\theta - \cos^4\theta) = \frac{16q^2a^4\omega^6Z_0}{15c^4\pi}
\end{aligned}$$

39. J, Problem 9.14.

Solution:

a) We first write the current density in cylindrical coordinates $\boldsymbol{J} = I_0\hat{\varphi}\delta(z)\delta(r-a)$. Then we need the Fourier transform:

$$\int d^3r' \boldsymbol{J} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'} = aI_0 \int_0^{2\pi} d\varphi' \hat{\varphi}' e^{-ika\sin\theta\cos(\varphi-\varphi')}$$
$$= aI_0 \left(-\hat{x} \int_0^{2\pi} d\varphi' \sin(\varphi'+\varphi) e^{-ika\sin\theta\cos(\varphi')} + \hat{y} \int_0^{2\pi} d\varphi' \cos(\varphi'+\varphi) e^{-ika\sin\theta\cos(\varphi')} \right)$$

Using the addition formulas for sine and cosine on the factor multiplying the exponential, the term with $\sin\varphi'$ is a total derivative that integrates to zero by periodicity. So only the $\cos\varphi'$ terms survive

$$\int d^3r' \boldsymbol{J} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'} = aI_0\hat{\varphi} \int_0^{2\pi} d\varphi' \cos\varphi' e^{-ika\sin\theta\cos\varphi'} = 2i\pi a I_0 \hat{\varphi} J_1(ka\sin\theta)$$

where we have recognized the integral as one of the representations of the Bessel function J_1. This Fourier transform enters the expressions for the fields in radiation zone:

$$\boldsymbol{H} \sim 2i\pi a I_0 \hat{\varphi} J_1(ka\sin\theta) i \frac{e^{ikr}}{4\pi r} k\hat{k}\times\hat{\varphi}, \qquad \boldsymbol{E} = -Z_0\hat{k}\times\boldsymbol{H}$$

Then

$$\frac{dP}{d\Omega} = \frac{r^2}{2}\hat{k}\text{Re } \boldsymbol{E}\times\boldsymbol{H}^* = \frac{Z_0 k^2 a^2 I_0^2}{8} J_1^2(ka\sin\theta) \tag{2}$$

where we used $(\hat{k}\times\hat{\varphi})^2 = 1$ because $\hat{k}$ is orthogonal to $\hat{\varphi}$.

b) Intuitively the lowest nonvanishing multipole is the magnetic dipole with moment $\boldsymbol{m} = \pi a^2 I_0$. In the lm basis this correponds to M_{1m}. We can confiem this by taking the $ka \ll 1$ approximation of the result of a) using $J_1(x) \sim x/2$ for small x,

$$\frac{dP}{d\Omega} \sim \frac{Z_0 k^2 a^2 I_0^2}{32}(ka\sin\theta)^2 = \frac{Z_0 k^4 (\pi a^2 I_0)^2}{32\pi^2}\sin^2\theta \tag{3}$$

which is the formula for maagnetic dipole radiation with $|\boldsymbol{m}| = \pi a^2 I_0$. We can also evaluate the moments M_{lm} from Eq. J(9.172).

$$\begin{aligned}
M_{lm} &= -\frac{1}{l+1}\int d^3r r^l Y^*_{lm}\nabla\cdot(\boldsymbol{r}\times\boldsymbol{J}) = -\frac{1}{l+1}\int d^3r(\boldsymbol{r}\times\nabla r^l Y_{lm})^*\cdot\boldsymbol{J} \\
&= -\frac{1}{l+1}\int d\varphi I_0 a^{l+1}\hat{\varphi}\cdot(i\boldsymbol{L}Y_{lm}(\pi/2,\varphi))^* \\
&= \frac{ia^{l+1}I_0}{l+1}\int d\varphi(-\sin\varphi L_x Y_{lm}(\pi/2,\varphi)+\cos\varphi L_y Y_{lm}(\pi/2,\varphi))^* \\
&= \frac{ia^{l+1}I_0}{2(l+1)}\delta_{m0}\int d\varphi(ie^{i\varphi}L_- Y_{l0}(\pi/2,\varphi) - ie^{-i\varphi}L_+ Y_{l0}(\pi/2,\varphi))^* \\
&= \frac{a^{l+1}I_0\sqrt{l(l+1)}}{2(l+1)}\delta_{m0}\int d\varphi(e^{i\varphi}Y_{l-1}(\pi/2,\varphi) - e^{-i\varphi}Y_{l1}(\pi/2,\varphi))^* \\
&= \frac{a^{l+1}I_0\sqrt{l(l+1)}}{2(l+1)}\delta_{m0}\int d\varphi(-e^{i\varphi}Y^*_{l1}(\pi/2,\varphi) - e^{-i\varphi}Y_{l1}(\pi/2,\varphi))^* \\
&= \frac{a^{l+1}I_0\sqrt{l(l+1)}}{2(l+1)}\delta_{m0}4\pi\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-1)!}{(l+1)!}}P^1_l(0) = \frac{a^{l+1}I_0}{(l+1)}\delta_{m0}2\pi\sqrt{\frac{2l+1}{4\pi}}P^1_l(0)
\end{aligned}$$

For $l = 1$ we see from (3.49), that $P^1_1(0) = -1$ so the lowest moment is $M_{1m} = \pi a^2 I_0\sqrt{3/4\pi} = \delta_{m0}|\boldsymbol{m}|\sqrt{3/4\pi}$ which is the correct value for a magnetic moment $\boldsymbol{m}$ in the lm basis.