Electromagnetic Theory II

Solution Set 11

Due: 7 April 2021

40. Consider a linear antenna of length d for which the current density is given to be $\boldsymbol{J} = I\hat{z}\delta(x)\delta(y)\sin(2\pi z/d)e^{-i\omega t}$ for $-d/2 < z < d/2$, so the current density vanishes at the endpoints and at the midpoint of the antenna.

a) Evaluate the Fourier transform $\int d^3x \boldsymbol{J} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}$ exactly and use it to determine the exact power angular distribution $dP/d\Omega$ for any k. Check that for wavelength $2\pi/k = d$ your answer simplifies to

$$\frac{dP}{d\Omega}_{k=2\pi/d} = \frac{Z_0 I^2}{8\pi^2}\frac{\sin^2[\pi\cos\theta]}{\sin^2\theta} \tag{1}$$

Solution:

$$\begin{aligned} I\hat{z}\int_{-d/2}^{d/2} dz \sin\frac{2\pi z}{d} e^{-ikz\cos\theta} &= I\hat{z}\frac{1}{2i}\left(\frac{e^{(2\pi/d-k\cos\theta)iz}}{2\pi i/d - ik\cos\theta} - \frac{e^{-(2\pi/d+k\cos\theta)iz}}{-2\pi i/d - ik\cos\theta}\right)\Bigg|_{-d/2}^{d/2} \\ &= I\hat{z}\frac{-i(4\pi/d)\sin[(kd/2)\cos\theta]}{-k^2\cos^2\theta + 4\pi^2/d^2} \end{aligned}$$

so the angular distribution becomes

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{Z_0 k^2 I^2 \sin^2\theta}{32\pi^2}\frac{(16\pi^2/d^2)\sin^2[(kd/2)\cos\theta]}{(-k^2\cos^2\theta + 4\pi^2/d^2)^2} \\ &\to \frac{Z_0 I^2\sin^2\theta}{8\pi^2}\frac{\sin^2[\pi\cos\theta]}{(1-\cos^2\theta)^2} = \frac{Z_0 I^2}{8\pi^2}\frac{\sin^2[\pi\cos\theta]}{\sin^2\theta} \end{aligned}$$

where the last line specializes to $k = 2\pi/d$.

b) Now apply our vector spherical harmonics method to this current density. Show that all the magnetic multipole coefficients $a^M_{lm} = 0$, and by expressing the electric multipole coefficients a^E_{lm} as an integral over z with $0 < z < d/2$ show that only the ones with l even and $m = 0$ are nnvanishing.

Solution: The exact multipole coefficients are a^E_{lm}, a^M_{lm}. Because $\boldsymbol{r}\times\boldsymbol{J} = 0$ for the current of part a), all the $a^M_{lm} = 0$. According to J (9.167)[1] the evaluation of a^E requires

$$
\begin{aligned}
c\rho &= \frac{c}{i\omega}\nabla\cdot\boldsymbol{J} = \frac{I}{ik}\frac{2\pi}{d}\cos\frac{2\pi z}{d}\delta(x)\delta(y) \\
\boldsymbol{r}\cdot\boldsymbol{J} &= Iz\sin\frac{2\pi z}{d}\delta(x)\delta(y) \\
a^E_{lm} &= \frac{k^2}{i\sqrt{l(l+1)}}\int dz Y^*_{lm}(\theta=0)\left(\frac{I}{ik}\frac{2\pi}{d}\cos\frac{2\pi z}{d}[xj_l(x)]'|_{x=k|z|} + ikj_l(k|z|)Iz\sin\frac{2\pi z}{d}\right)
\end{aligned}
$$

In this formula $Y^*_{lm} = \delta_{m0}\sqrt{(2l+1)/4\pi}$ is evaluated at $\theta = 0$ for $z > 0$ and $Y^*_{lm}(\theta = \pi) = (-)^l\delta_{m0}\sqrt{(2l+1/4)\pi}$ at $\theta = \pi$ for $z < 0$. Since the rest of the integrand is even in z, we may rewrite the formula as

$$
\begin{aligned}
a^E_{lm} &= \delta_{m0}(1+(-)^l)\frac{k^2}{i}\sqrt{\frac{2l+1}{4\pi l(l+1)}}\int_0^{d/2} dz \\
&\left(\frac{I}{ik}\frac{2\pi}{d}\cos\frac{2\pi z}{d}[xj_l(x)]'|_{x=kz} + ikj_l(kz)Iz\sin\frac{2\pi z}{d}\right)
\end{aligned}
$$

c) For the special value of $k = 2\pi/d$, notice that the integrand of part b) is a total derivative, and so evaluate all of the nonvanishing a^E exactly.

Solution: Setting $k = 2\pi/d$ shows that the integrand is a total derivative

$$
\begin{aligned}
a^E(l,0) &= (1+(-)^l)\frac{4\pi^2}{id^2}\sqrt{\frac{2l+1}{4\pi l(l+1)}}\int_0^{d/2} dz\frac{d}{dz}\left(\frac{I}{i}\cos\frac{2\pi z}{d}zj_l(kz)\right) \\
&= (1+(-)^l)\frac{4\pi^2}{d^2}\sqrt{\frac{2l+1}{4\pi l(l+1)}}\left(\frac{Id}{2}j_l(\pi)\right) = (1+(-)^l)\frac{2\pi^2 Ij_l(\pi)}{d}\sqrt{\frac{2l+1}{4\pi l(l+1)}}
\end{aligned}
$$

So the only nonvanishing multipoles are electric and even l. The lowest is the quadrupole $l = 2$

$$
a^E(2,0) = \frac{4\pi^2 Ij_2(\pi)}{d}\sqrt{\frac{5}{24\pi}} = \frac{12I}{d}\sqrt{\frac{5}{24\pi}} = \frac{I}{d}\sqrt{\frac{30}{\pi}} \tag{2}
$$

d) For $k = 2\pi/d$, evaluate the contribution of the lowest nonvanishing a^E to $dP/d\Omega$ and compare to the exact result of part a). Plot both distributions on the same graph.

[1] Starting from Eq.(733) of our lecture notes, the following rearrangement leads to the same result

$$
\boldsymbol{L}\cdot(\nabla\times\boldsymbol{J}) = i[(\nabla\times\boldsymbol{r})\times\nabla]\cdot\boldsymbol{J} = i[\nabla_i\boldsymbol{r}\nabla_i - \nabla(r_i\nabla_i)]\cdot\boldsymbol{J} = i[\nabla^2(\boldsymbol{r}\cdot\boldsymbol{J}) - (2+r_i\nabla_i)(\nabla\cdot\boldsymbol{J})]
$$

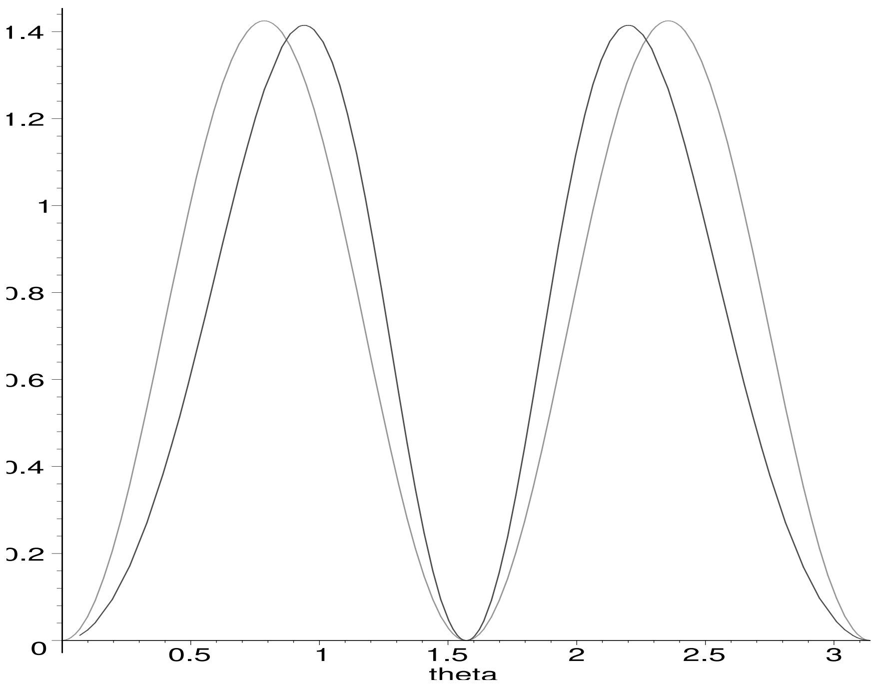
Solution: The contribution of this lowest multipole to the power per solid angle is (see J, (9.150)

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{Z_0}{2k^2}|a^E(2,0)|^2|\boldsymbol{X}_{20}\times\boldsymbol{n}|^2 \to \frac{Z_0 d^2}{8\pi^2}|a^E(2,0)|^2|\boldsymbol{X}_{20}\times\boldsymbol{n}|^2 \\
|\boldsymbol{X}_{20}\times\boldsymbol{n}|^2 &= |\boldsymbol{X}_{20}|^2 - |\boldsymbol{n}\cdot\boldsymbol{X}_{20}|^2 = |\boldsymbol{X}_{20}|^2 = \frac{1}{6}|\boldsymbol{L}Y_{20}|^2 \\
L_z Y_{20} &= 0, \qquad L_x Y_{20} = \frac{1}{2}(L_+ + L_-)Y_{20} = \frac{\sqrt{6}}{2}(Y_{21}+Y_{2,-1}) = -i\sqrt{\frac{45}{4\pi}}\sin\theta\cos\theta\sin\varphi \\
L_y Y_{20} &= \frac{1}{2i}(L_+ - L_-)Y_{20} = \frac{\sqrt{6}}{2i}(Y_{21}-Y_{2,-1}) = i\sqrt{\frac{45}{4\pi}}\sin\theta\cos\theta\cos\varphi \\
|\boldsymbol{X}_{20}|^2 &= \frac{15}{8\pi}\sin^2\theta\cos^2\theta \qquad (3)
\end{aligned}$$

This angular distribution is to be compared to the exact angular dependence

$$\sin^2(\pi\cos\theta)/\sin^2\theta$$

The behavior near $\pi/2$, when $\cos\theta \approx 0$ is similar. It is also similar behavior near $\theta \sim 0, \pi$.



The curve skewed more toward $\theta = \pi/2$ is the exact one.

41. The electric field of a plane wave $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ is perpendicular to the propagation direction $\boldsymbol{k}$. The polarization vector $\boldsymbol{\varepsilon}$ is by definition a unit vector parallel to the electric field. We have found it useful to describe the fields with complex vectors whose real parts are the physical

fields. In this description polarization vectors can be complex when they describe elliptic polarization, in which case we adopt the normalization $\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon} = 1$. Complex vectors belong to a three dimensional complex vector space, and it is convenient to pick an orthonormal basis. Let $\boldsymbol{n} = \hat{k}$ be the unit vector parallel to $\boldsymbol{k}$. Then choose $\boldsymbol{e}_3 = \boldsymbol{n}$, $\boldsymbol{e}_1 = \boldsymbol{\varepsilon}$, and $\boldsymbol{e}_2 = \boldsymbol{n} \times \boldsymbol{\varepsilon}^*$.

a) Show that the three $\boldsymbol{e}_a$ are orthonormal in the sense that $\boldsymbol{e}_a^* \cdot \boldsymbol{e}_b = \delta_{ab}$, and prove the completeness relation $\sum_{a=1}^3 e_a^i e_a^{j*} = \delta_{ij}$. Note that the completeness relation can be rearranged as

$$\sum_{a=1}^{2} e_a^i e_a^{j*} = \delta_{ij} - e_3^i e_3^{j*} = \delta_{ij} - n^i n^j$$

which is useful to sum cross sections over polarizations, $\sum_{\text{pol}} \epsilon^i \epsilon^{j*} = \delta_{ij} - n^i n^j$.

Solution: $\boldsymbol{\varepsilon}$ and $\boldsymbol{n}$ are by definition orthogonal and normalized to unity. $\boldsymbol{n} \times \boldsymbol{\varepsilon}^*$ is orthogonal to $\boldsymbol{n}$ because $\boldsymbol{n} \cdot (\boldsymbol{n} \times \boldsymbol{\varepsilon}^*) = (\boldsymbol{n} \times \boldsymbol{n}) \cdot \boldsymbol{\varepsilon}^* = 0$. Similarly it is orthogonal to $\boldsymbol{\varepsilon}$ because $(\boldsymbol{n} \times \boldsymbol{\varepsilon}^*) \cdot \boldsymbol{\varepsilon}^* = \boldsymbol{n} \cdot (\boldsymbol{\varepsilon}^* \times \boldsymbol{\varepsilon}^*) = 0$. Here it is crucial that the scalar product of two complex vectors is the complex conjugate of one dotted into the other. Checking the norm $(\boldsymbol{n} \times \boldsymbol{\varepsilon}^*) \cdot (\boldsymbol{n} \times \boldsymbol{\varepsilon}) = \boldsymbol{n} \cdot (\boldsymbol{\varepsilon}^* \times (\boldsymbol{n} \times \boldsymbol{\varepsilon})) = \boldsymbol{n} \cdot \boldsymbol{n} \boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon} = 1$ completes the proof of orthonormality. Completeness follows if the three $\boldsymbol{e}_a$ are linearly independent. This follows because if $\sum c_a \boldsymbol{e}_a = 0$, taking the scalar product of this relation with $\boldsymbol{e}_b$ shows that $c_b = 0$ for all b.

b) We can expand any real unit vector in three space $\boldsymbol{v} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z}$ in the new basis $\boldsymbol{v} = \sum_a V_a \boldsymbol{e}_a$. Show that $\sum_a |V_a|^2 = 1$

Solution: By orthonormality it follows that $\boldsymbol{v}^* \cdot \boldsymbol{v} = \sum_a |V_a|^2$. But since $\boldsymbol{v}$ is real $\boldsymbol{v}^* \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{v} = v_x^2 + v_y^2 + v_z^2 = 1$, because $\boldsymbol{v}$ is a real unit vector. This completes the proof.

c) Prove the identity

$$|\hat{r} \cdot \boldsymbol{n}|^2 + |\hat{r} \cdot \boldsymbol{\epsilon}|^2 + |\hat{r} \cdot (\boldsymbol{n} \times \boldsymbol{\varepsilon})|^2 = 1$$

where $\hat{r}$ is the radial unit vector.

Solution: Expanding $\hat{r} = \sum_a V_a \boldsymbol{e}_a$, orthonormality shows that $V_3 = \hat{r} \cdot \boldsymbol{n}$, $V_1 = \hat{r} \cdot \boldsymbol{\varepsilon}^*$, and $V_3 = \hat{r} \cdot (\boldsymbol{n} \times \boldsymbol{\varepsilon})$. Then part b) shows that

$$1 = |\hat{r} \cdot \boldsymbol{n}|^2 + |\hat{r} \cdot \boldsymbol{\epsilon}^*|^2 + |\hat{r} \cdot (\boldsymbol{n} \times \boldsymbol{\varepsilon})|^2 = |\hat{r} \cdot \boldsymbol{n}|^2 + |\hat{r} \cdot \boldsymbol{\epsilon}|^2 + |\hat{r} \cdot (\boldsymbol{n} \times \boldsymbol{\varepsilon})|^2$$

42. J, Problem 10.1. Hint: apply the results of problem 41 to the result in Eq. (10.14).
Solution:

a) We write (10.14) as

$$\frac{d\sigma}{d\Omega} = k^4a^6|\boldsymbol{\varepsilon}^*\cdot(\boldsymbol{\varepsilon}_0+\boldsymbol{n}\times(\boldsymbol{n}_0\times\boldsymbol{\varepsilon}_0)/2)|^2 \equiv |\boldsymbol{\varepsilon}^*\cdot\boldsymbol{M}|^2 \tag{4}$$

By problem 41b the sum over polarizations is

$$\begin{aligned}\sum \boldsymbol{\varepsilon}\cdot\boldsymbol{M}\boldsymbol{\varepsilon}^*\cdot\boldsymbol{M}^* &= \boldsymbol{M}\cdot\boldsymbol{M}^* - |\boldsymbol{n}\cdot\boldsymbol{M}|^2 \\ &= k^4a^6\left(\boldsymbol{\varepsilon}_0^*\cdot\boldsymbol{\varepsilon}(1-\boldsymbol{n}\cdot\boldsymbol{n}_0/2)^2+\frac{1}{4}|\boldsymbol{n}\cdot\boldsymbol{\varepsilon}_0|^2-|\boldsymbol{n}\cdot\boldsymbol{\varepsilon}_0|^2\right) \\ &= k^4a^6\left(1-\boldsymbol{n}\cdot\boldsymbol{n}_0+\frac{1}{4}((\boldsymbol{n}\cdot\boldsymbol{n}_0)^2+|\boldsymbol{n}\cdot\boldsymbol{\varepsilon}_0|^2)-|\boldsymbol{n}\cdot\boldsymbol{\varepsilon}_0|^2\right) \\ &= k^4a^6\left(\frac{5}{4}-\boldsymbol{n}\cdot\boldsymbol{n}_0-\frac{1}{4}|\boldsymbol{n}\cdot(\boldsymbol{n}_0\times\boldsymbol{\varepsilon}_0)|^2-|\boldsymbol{n}\cdot\boldsymbol{\varepsilon}_0|^2\right) \qquad (5)\end{aligned}$$

by problem 41c.

b) Take $\boldsymbol{n}_0 = \hat{z}$ and $\boldsymbol{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Then $\boldsymbol{n}\cdot\boldsymbol{\varepsilon}_0 = \sin\theta\cos\varphi$ and $\boldsymbol{n}\cdot(\boldsymbol{n}_0\times\boldsymbol{\varepsilon}_0) = \sin\theta\sin\varphi$ so that

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= k^4a^6\left(\frac{5}{4}-\cos\theta-\frac{1}{4}\sin^2\theta\sin^2\varphi-\sin^2\theta\cos^2\varphi\right) \\ &= k^4a^6\left(\frac{5}{4}-\cos\theta-\frac{1}{4}\sin^2\theta\frac{1-\cos 2\varphi}{2}-\sin^2\theta\frac{1+\cos 2\varphi}{2}\right) \\ &= k^4a^6\left(\frac{5}{8}(1+\cos^2\theta)-\cos\theta-\frac{3}{8}\sin^2\theta\cos 2\varphi\right)\end{aligned}$$

c) Evaluating b) for $\theta = \pi/2, \varphi = 0$ gives $k^4a^6/4$ and for $\theta = \pi/2, \varphi = \pi/2$ gives k^4a^6. Thus the ratio is 1/4. To understand this recall that the dipole approximation to the electric field is proportional to $\boldsymbol{n}\times(\boldsymbol{n}\times(\boldsymbol{p}-\boldsymbol{n}\times\boldsymbol{m}/c))$. Taking $\boldsymbol{k}_0$ in the z-direction and $\boldsymbol{\varepsilon}_0 = \hat{x}$, the induced dipole moments in a perfectly conducting sphere are

$$\begin{aligned}\boldsymbol{p} &= 4\pi a^3\epsilon_0\hat{x}, \qquad \boldsymbol{m} = -\frac{2\pi a^3}{Z_0}\hat{z}\times\boldsymbol{\varepsilon}_0 = -2\pi a^3\epsilon_0 c\hat{y} \\ \boldsymbol{p}-\boldsymbol{n}\times\boldsymbol{m}/c &= 2\pi a^3\epsilon_0(2\hat{x}+\boldsymbol{n}\times\hat{y})\end{aligned}$$

When $\theta = \pi/2, \varphi = 0$, $\boldsymbol{n} = \hat{x}$ and only the $\boldsymbol{m}$ term contributes. On the other hand when $\varphi = \pi/2$, $\boldsymbol{n} = \hat{y}$ and only the $\boldsymbol{p}$ term contributes. The electric moment contributes twice as much as the magnetic moment to the electric field, so its contribution to the scattering cross section is 4 times as much. this explains the ratio of 4.

43. J, Problem 10.2. Again it is sufficient to work with Eq. (10.14).

Solution: Let us write out the polarization vector in Cartesian components assuming $\boldsymbol{k}_0 = k\hat{z}$:

$$\begin{aligned}
\boldsymbol{\varepsilon}_0 &= \frac{1}{\sqrt{1+r^2}}(\boldsymbol{\varepsilon}_+ + re^{i\alpha}\boldsymbol{\varepsilon}_-) = \frac{1}{\sqrt{2(1+r^2)}}[\hat{x}(1+re^{i\alpha}) + i\hat{y}(1-re^{i\alpha})] \\
\boldsymbol{n}_0 \times \boldsymbol{\varepsilon}_0 &= \frac{1}{\sqrt{2(1+r^2)}}[\hat{y}(1+re^{i\alpha}) - i\hat{x}(1-re^{i\alpha})] \\
\boldsymbol{n} \cdot \boldsymbol{\varepsilon}_0 &= \frac{\sin\theta}{\sqrt{2(1+r^2)}}(e^{i\varphi} + re^{i(\alpha-\varphi)}), \qquad \boldsymbol{n} \cdot (\boldsymbol{n}_0 \times \boldsymbol{\varepsilon}_0) = -\frac{i\sin\theta}{\sqrt{2(1+r^2)}}(e^{i\varphi} - re^{i(\alpha-\varphi)})
\end{aligned}$$

To calculate the cross section from (10.1 a) we need

$$|e^{i\varphi} \pm re^{i(\alpha-\varphi}|^2 = 1 + r^2 \pm 2r\cos(2\varphi - \alpha) \tag{6}$$

Then

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= k^4a^6\left[\frac{5}{4} - \frac{\sin^2\theta}{2(1+r^2)}(1+r^2+2r\cos(2\varphi-\alpha)) - \frac{\sin^2\theta}{8(1+r^2)}(1+r^2-2r\cos(2\varphi-\alpha)) - \cos\theta\right] \\
&= k^4a^6\left[\frac{5}{4} - \frac{5}{8}\sin^2\theta - \frac{3r\sin^2\theta}{4(1+r^2)}\cos(2\varphi-\alpha) - \cos\theta\right] \\
&= k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \frac{3r\sin^2\theta}{4(1+r^2)}\cos(2\varphi-\alpha) - \cos\theta\right]
\end{aligned}$$

The case of linear polarization is $r = 1, \alpha = 0$. Evaluating for these values the result reduces to that of (10.1b).