Electromagnetic Theory II

Solution Set 12

Due: 14 April 2021

44. J, Problem 10.3. The absorption cross section asked for in part b) is the power absorbed by the scatterer divided by the incident flux. For large conductivity the power absorbed per unit area is $|\boldsymbol{H}_\parallel|^2/2\sigma\delta$ given by J, Eq.(8.12) or Eq. (606) in Chapter 10 0f our notes. Here $\boldsymbol{H}_\parallel$ is the tangential magnetic field at the surface of the conductor, which you find in part a).

Solution:

a) Since $kR \ll 1$ the sphere sees an approximately uniform field that is changing very slowly. So we first solve the magnetostatic problem of a perfectly conducting sphere in a uniform field $\boldsymbol{H} = H_0\hat{z}$ by setting $\boldsymbol{H}_S = -\nabla\phi$ with

$$\phi = -H_0 z + \frac{A}{r^2}\cos\theta = \left(-H_0 r + \frac{A}{r^2}\right)\cos\theta \tag{1}$$

Perfect conductor boundary conditions are $\hat{r}\cdot\boldsymbol{H} = 0$, or $A = -H_0R^3/2$, leading to

$$\boldsymbol{H} \;=\; H_0\left(\hat{z} - \frac{3R^3 z\hat{r} - R^3 r\hat{z}}{2r^4}\right) \to \frac{3H_0}{2}(\hat{z} - \hat{r}\cos\theta), \qquad r\to R \tag{2}$$

Of course this static solution doesn't satisfy Maxwell's equations for $\omega \neq 0$, but it should give the $\omega \to 0$ limit of the exact solution. Indeed, in our exact scattering solution for a perfect sphere, the leading term in $\boldsymbol{H}$ is the a^M_{1m} term in the vector spherical harmonic expansion (taking linear polarization $\boldsymbol{E}_0 = Z_0H_0\hat{x}$, $\boldsymbol{H}_0 = H_0\hat{y}$, and setting $r \to R$):

$$\begin{aligned}
\boldsymbol{H} &\sim \frac{-i}{k}a^M_{1m}\nabla\times(g_1(kr)\boldsymbol{X}_{1m} \to -ia^M_{1m}g_l'(kR)\hat{r}\times\boldsymbol{X}_{1m} \\
&\sim H_0\sqrt{3\pi}\left(j_1'(kR) - \frac{j_1(kR)}{h_1^{(1)}(kR)}h_1^{(1)\prime}(kR)\right)\hat{r}\times(\boldsymbol{X}_{1,-1} + \boldsymbol{X}_{1,1}) \\
&\sim \frac{3H_0}{2}\left(j_1'(kR) - \frac{j_1(kR)}{h_1^{(1)}(kR)}h_1^{(1)\prime}(kR)\right)\hat{r}\times(\hat{y}\times\hat{r})
\end{aligned}$$

where we used $g_l(kR) = 0$. The quantity in parentheses tends to 1 as $kR \to 0$, so this gives

$$\boldsymbol{H} \sim \frac{3H_0}{2}(\hat{y} - \hat{r}\sin\theta\sin\varphi) \tag{3}$$

which matches the static calculation for a uniform magnetic field in the y-direction.

b) The power absorbed by the sphere is given, in the case that $\delta \ll R$ by integrating formula (8.12)

$$\frac{dP_{\rm loss}}{dA} = \frac{\mu_c\omega\delta}{4}|\boldsymbol{H}_\parallel|^2 \tag{4}$$

over the surface of the sphere:

$$P_{\rm loss} = \frac{\mu_c\omega\delta R^2 H_0^2}{4}\int d\Omega \frac{9}{4}\sin^2\theta = \frac{3\pi}{2}\mu_c\omega\delta R^2 H_0^2 \tag{5}$$

The absorption cross section is obtained by dividing by the incident flux $Z_0 H_0^2/2$:

$$\sigma_{\rm abs} = 3\pi R^2 \frac{\mu_c\omega\delta}{Z_0} = \frac{3\pi R^2}{Z_0}\sqrt{\frac{2\mu_c\omega}{\sigma}} \tag{6}$$

which shows the $\sqrt{\omega}$ behavior.

45. In our class discussion of scattering of electromagnetic waves from a perfectly conducting sphere we showed that the long wavelength limit of the exact scattering amplitude gave the behavior

$$\boldsymbol{f} \sim -\sqrt{\frac{\pi}{3}}k^2R^3\left[(\epsilon_{0+}\boldsymbol{X}_{1-1} + \epsilon_{0-}\boldsymbol{X}_{11}) + (2i)\hat{k}\times(\epsilon_{0+}\boldsymbol{X}_{1-1} - \epsilon_{0-}\boldsymbol{X}_{11})\right]. \tag{7}$$

Using the explicit forms for the vector spherical harmonics, show that this reproduces the forms

$$\begin{aligned}\boldsymbol{f} &= -\frac{k^2R^3}{2}\left[-\hat{k}\times(\hat{k}_0\times\boldsymbol{\varepsilon}_0) + 2\hat{k}\times(\hat{k}\times\boldsymbol{\varepsilon}_0)\right]\\ \boldsymbol{\varepsilon}^*\cdot\boldsymbol{f} &= k^2R^3\left[-\frac{1}{2}(\hat{k}\times\boldsymbol{\varepsilon}^*)\cdot(\hat{k}_0\times\boldsymbol{\varepsilon}_0) + \boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right]\end{aligned} \tag{8}$$

obtained in our earlier discussion of the long wavelength approximation.

Solution: We first work out the explicit forms for $\boldsymbol{X}_{1,\pm1} = \boldsymbol{L}Y_{1,\pm1}/\sqrt{2}$, recalling formulas from angular momentum in quantum mechanics:

$$\begin{aligned}L_zY_{1\pm1} &= \pm Y_{1\pm1}, \qquad (L_x \pm iL_y)Y_{1m} = \sqrt{1(1+1)-m(m\pm1)}Y_{1,m\pm1}\\ X^z_{1\pm1} &= \pm\frac{1}{\sqrt{2}}Y_{1\pm1}, \qquad X^x_{11} \pm iX^y_{11} = \begin{cases}0\\ Y_{10}\end{cases}, \qquad X^x_{1,-1} \pm iX^y_{1-1} = \begin{cases}Y_{10}\\ 0\end{cases}\end{aligned} \tag{9}$$

Now

$$
\begin{aligned}
Y_{1,\pm1} &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi}, \qquad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \\
\boldsymbol{X}_{11} &= \left(\frac{Y_{10}}{2}, \frac{iY_{10}}{2}, \frac{Y_{11}}{\sqrt{2}}\right) = \sqrt{\frac{3}{16\pi}}\left(\cos\theta, i\cos\theta, -\sin\theta e^{i\varphi}\right) \\
\boldsymbol{X}_{1,-1} &= \left(\frac{Y_{10}}{2}, -\frac{iY_{10}}{2}, -\frac{Y_{1,-1}}{\sqrt{2}}\right) = \sqrt{\frac{3}{16\pi}}\left(\cos\theta, -i\cos\theta, -\sin\theta e^{-i\varphi}\right) \\
\varepsilon_{0+}\boldsymbol{X}_{1-1} + \varepsilon_{0-}\boldsymbol{X}_{11} &= \sqrt{\frac{3}{4\pi}}\left[\varepsilon_{0x}\left(\cos\theta, 0, -\sin\theta\sin\varphi\right) + i\varepsilon_{0y}\left(0, -i\cos\theta, i\sin\theta\cos\varphi\right)\right] \\
\varepsilon_{0+}\boldsymbol{X}_{1-1} - \varepsilon_{0-}\boldsymbol{X}_{11} &= \sqrt{\frac{3}{4\pi}}\left[i\varepsilon_{0y}\left(\cos\theta, 0, -\sin\theta\sin\varphi\right) + \varepsilon_{0x}\left(0, -i\cos\theta, i\sin\theta\cos\varphi\right)\right]
\end{aligned}
$$

For comparison we examine

$$
\begin{aligned}
\hat{k}\times\boldsymbol{\varepsilon}_0 &= \varepsilon_{0x}(-\hat{z}\sin\theta\sin\varphi + \hat{y}\cos\theta) + \varepsilon_{0y}(\hat{z}\sin\theta\cos\varphi - \hat{x}\cos\theta) \\
\hat{k}\times(\hat{z}\times\boldsymbol{\varepsilon}_0) &= -\varepsilon_{0y}(-\hat{z}\sin\theta\sin\varphi + \hat{y}\cos\theta) + \varepsilon_{0x}(\hat{z}\sin\theta\cos\varphi - \hat{x}\cos\theta)
\end{aligned}
$$

from which we see that

$$
\begin{aligned}
\varepsilon_{0+}\boldsymbol{X}_{1-1} + \varepsilon_{0-}\boldsymbol{X}_{11} &= -\sqrt{\frac{3}{4\pi}}\hat{k}\times(\hat{z}\times\boldsymbol{\varepsilon}_0), \qquad \varepsilon_{0+}\boldsymbol{X}_{1-1} - \varepsilon_{0-}\boldsymbol{X}_{11} = -i\sqrt{\frac{3}{4\pi}}\hat{k}\times\boldsymbol{\varepsilon}_0 \\
\boldsymbol{f} &\sim \frac{k^2R^3}{2}\left[\hat{k}\times(\hat{z}\times\boldsymbol{\varepsilon}_0) - 2\hat{k}\times\left(\hat{k}\times\boldsymbol{\varepsilon}_0\right)\right] \\
\boldsymbol{\varepsilon}^*\cdot\boldsymbol{f} &\sim \frac{k^2R^3}{2}\left[(\boldsymbol{\varepsilon}^*\times\hat{k})\cdot(\hat{z}\times\boldsymbol{\varepsilon}_0) + 2\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right] \\
&= k^2R^3\left[-\frac{1}{2}(\hat{k}\times\boldsymbol{\varepsilon}^*)\cdot(\hat{z}\times\boldsymbol{\varepsilon}_0) + \boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right]
\end{aligned}
$$

46. J, Problem 10.7

Solution:

a) In the region outside the sphere, we use the same notation as in class for the perfectly conducting sphere, $a^{E,M}_{0lm}$ are the coefficients for the incident plane wave, for which the $j_l(kr)$ are the appropriate spherical Bessel functions and $a^{E,M}_{Slm}$ for the scattered wave for which $h^{(1)}_l(kr)$ are appropriate. For the inside of the sphere the functions $j_l(k'r)$, with $k' = k\sqrt{\epsilon/\epsilon_0}$, are the appropriate Bessel functions. We use the notation $b^{E,M}_{lm}$ for their coefficients. These coefficients will be used as is for the $\boldsymbol{H}$ fields, and so for the $\boldsymbol{E}$ fields they must be multiplied by either $Z_0 = \sqrt{\mu_0/\epsilon_0}$ for the exterior solution and by

$Z = Z_0\sqrt{\epsilon_0/\epsilon}$ for the interior solution. Then the matching conditions are as follows:

$$
\begin{aligned}
b^M_{lm} j_l(k'a) &= \frac{k'}{k}[a^M_{0lm} j_l(ka) + a^M_{Slm} h^{(1)}_l(ka)] \\
&= \sqrt{\frac{\epsilon}{\epsilon_0}}[a^M_{0lm} j_l(ka) + a^M_{Slm} h^{(1)}_l(ka)], \qquad \text{Continuity of } \boldsymbol{r}\cdot\boldsymbol{B} \\
b^E_{lm} j_l(k'a) &= a^E_{0lm} j_l(ka) + a^E_{Slm} h^{(1)}_l(ka), \qquad \text{Continuity of } \boldsymbol{r}\cdot\boldsymbol{D} \\
b^M_{lm}(rj_l(k'r))' &= \sqrt{\frac{\epsilon}{\epsilon_0}}\left[a^M_{0lm}(rj_l(kr))' + a^M_{Slm}(rh^{(1)}_l(kr))'\right], \qquad \text{Continuity of } \boldsymbol{r}\times\boldsymbol{H} \\
b^E_{lm}(rj_l(k'r))' &= \frac{\epsilon}{\epsilon_0}\left[a^E_{0lm}(rj_l(kr))' + a^E_{Slm}(rh^{(1)}_l(kr))'\right], \qquad \text{Continuity of } \boldsymbol{r}\times\boldsymbol{E}
\end{aligned}
$$

Some remarks: The conditions on the first two lines are necessary for the last two lines to guarantee the continuity of the respective tangential fields. The $()'$ denotes $\partial/\partial r$ applied to the quantity in parentheses after which r is set equal to a. These four equations determine the b's and a_S's in terms of the a_0's which are already known. We need the a_S's to get the scattering amplitudes:

$$
\begin{aligned}
a^E_{Slm} &= a^E_{0lm}\frac{\epsilon_0 j_l(ka)(rj_l(k'r))' - \epsilon j_l(k'a)(rj_l(kr))'}{\epsilon j_l(k'a)(rh^{(1)}_l(kr))' - \epsilon_0 h^{(1)}_l(ka)(rj_l(k'r))'} \equiv a^E_{0lm}\frac{\beta_l}{2} \\
a^M_{Slm} &= a^M_{0lm}\frac{j_l(ka)(rj_l(k'r))' - j_l(k'a)(rj_l(kr))'}{j_l(k'a)(rh^{(1)}_l(kr))' - h^{(1)}_l(ka)(rj_l(k'r))'} \equiv a^E_{0lm}\frac{\alpha_l}{2}
\end{aligned}
$$

Using $2j_l(ka) = h^{(1)}(ka) + h^{(2)}(ka)$, $\alpha_l + 1 \equiv e^{2i\delta_l}$ and $\beta_l + 1 \equiv e^{2i\delta'_l}$ can be shown to be phases, that is δ_l, δ'_l are real.

b) To study the long wavelength limit $ka \ll 1$ we use

$$
\begin{aligned}
j_l(x) &\sim \frac{x^l}{(2l+1)!!}, \qquad (xj_l(x))' \sim (l+1)\frac{x^l}{(2l+1)!!} \\
h^{(1)}_l(x) &\sim -i\frac{(2l-1)!!}{x^{l+1}}, \qquad (xh^{(1)}_l(x))' \sim il\frac{(2l-1)!!}{x^{l+1}} \\
\beta_l &\sim -2i\frac{(l+1)(ka)^{2l+1}}{(2l+1)!!(2l-1)!!}\frac{\epsilon_0 - \epsilon}{l\epsilon + (l+1)\epsilon_0}
\end{aligned} \tag{10}
$$

Of these the leading contribution is $l = 1$. Inspection of the numerator of α_l shows that the leading contribution of order $(ka)^{2l+1}$ cancels so the leading contribution starts at order $(ka)^{2l+3}$ so all the α_l's are subleading. Then the long wavelength limit of the scattering amplitude is

$$
\begin{aligned}
\boldsymbol{f} &\sim \frac{\sqrt{3\pi}}{2k}\beta_1\hat{k}\times(\epsilon_{0+}\boldsymbol{X}_{1,-1} - \epsilon_{0-}\boldsymbol{X}_{1,+1}) = -i\frac{3}{4k}\beta_1\hat{k}\times(\hat{k}\times\boldsymbol{\epsilon}_0) \\
&\sim k^2a^3\frac{\epsilon_0 - \epsilon}{\epsilon + 2\epsilon_0}\hat{k}\times(\hat{k}\times\boldsymbol{\epsilon}_0) \\
\frac{d\sigma}{d\Omega} &= |\boldsymbol{\epsilon}^*\cdot\boldsymbol{f}| \sim k^4a^6|\boldsymbol{\epsilon}^*\cdot\boldsymbol{\epsilon}_0|^2\frac{|\epsilon_0 - \epsilon|^2}{|\epsilon + 2\epsilon_0|^2}
\end{aligned} \tag{11}
$$

in agreement with J, (10.6).

c) To discus $\epsilon \to \infty$ we note that $k'a = ka\sqrt{\epsilon/\epsilon_0}$ also gets large. Thus we need the large argument limits of the Bessel functions

$$\begin{aligned} j_l(x') &\sim \frac{1}{x'}\sin(x' - l\pi/2), \quad (x' j_l(x'))' \sim \cos(x' - l\pi/2) \\ h_l^{(1)}(x') &\sim \frac{(-i)^{l+1}}{x'}e^{ix'}, \quad (x' h_l^{(1)}(x'))' \sim (-i)^l e^{ix'} \end{aligned} \tag{12}$$

Thus we see that $j_l(x') = O(\epsilon^{-1/2})(x' j_l(x'))'$. In the expression for α_l we see that this implies that the derivative terms dominate numerator and denominator and we find $\alpha_l \sim -2j_l(ka)/h_l^{(1)}(ka)$. However in the expression for β_l these derivative terms are multiplied by a factor ϵ^{-1} which suppresses them so we find $\beta_l \sim -2(rj_l(kr))'/(rh_l^{(1)}(kr))'$. Both of these results agree precisely with the magnetic and electric coefficients of the perfectly conducting sphere.

47. J, Problem 10.9. The lowest order approximation asked for in part a) is simply the Born approximation. The integral over angles to form the total cross section in part b) is best done by changing variables from θ to q, where $q^2 = (\boldsymbol{k} - k\hat{z})^2 = 2k^2(1 - \cos\theta)$. Since the result of part a) is simply the $ka \to \infty$ limit of that in part b), it suffices to only do the calculation in part b).

Solution:

a) The lowest order (i.e. Born approximation) to the scattering amplitude is given by

$$\begin{aligned} \boldsymbol{\epsilon}^* \cdot \boldsymbol{f} &= \boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0 \boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0 \frac{k^2}{4\pi}(\epsilon_r - 1)\int d^3x e^{i\boldsymbol{q}\cdot\boldsymbol{r}} \\ &= \boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0 \frac{k^2}{2}(\epsilon_r - 1)\int_0^a r^2 dr d\cos\theta e^{iqa\cos\theta} \\ &= \boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0 k^2(\epsilon_r - 1)\frac{1}{q}\int_0^a r dr \sin qr = \boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0(\epsilon_r - 1)\frac{k^2}{q^3}(\sin qa - qa\cos qa) \end{aligned}$$

where $\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{k}_0$. Then the differential cross section summed over final polarizations and averaged over intial polarizations is

$$\frac{d\sigma}{d\Omega} = (\epsilon_r - 1)^2\frac{k^4}{q^6}(\sin qa - qa\cos qa)^2\frac{1 + \cos^2\theta}{2}$$

We need $q^2 = (\boldsymbol{k} - \boldsymbol{k}_0)^2 = 2k^2(1 - \cos\theta)$, so $q \to 0$ in the forward direction. When $ka \gg 1$, finite scattering angle corresponds to $qa \gg 1$ and then $d\sigma/d\Omega = O(a^2k^4/q^4)$ compared to $O(a^2(ka)^4$ or larger when $qa \leq O(1)$, when $\theta \sim 0$. Clearly there is a sharp peak when $q \ll k$, i.e. when $\theta \to 0$. To calculate the total cross section, it is convenient to change integration variables from θ to q, $\cos\theta = 1 - q^2/2k^2$ and $d\Omega = d\varphi d(\cos\theta) = -d\varphi q dq/k^2$,

$$\begin{aligned}\sigma &= 2\pi k^2 a^4 (\varepsilon_r - 1)^2 \int_0^{2ka} \frac{dx}{x^5} (\sin x - x\cos x)^2 \left(1 - \frac{x^2}{2(ka)^2} + \frac{x^4}{8(ka)^4}\right) \\ &\to 2\pi k^2 a^4 (\varepsilon_r - 1)^2 \int_0^{\infty} \frac{dx}{x^5} (\sin x - x\cos x)^2\end{aligned}$$

for $ka \to \infty$. As we shall see from part b), the integral is just 1/4 yielding the desired result.

b) Setting $z = 2ka$ the integral required at finite ka is

$$I(z) = \int_0^z \frac{dx}{x^5} (\sin x - x\cos x)^2 \left(1 - 2\frac{x^2}{z^2} + 2\frac{x^4}{z^4}\right)$$

so we need the intgrals

$$I_n = \int_0^z \frac{dx}{x^n} \left(\frac{1-\cos 2x}{2} - x\sin 2x + x^2 \frac{1+\cos 2x}{2}\right), \qquad \text{for } n = 1, 3, 5. \tag{13}$$

The integrals for $n > 1$ can be reduced by a sequence of integrations by parts:

$$\begin{aligned}I_1 &= \int_0^z dx \left(\frac{1-\cos 2x}{2x} - \sin 2x + x\frac{1+\cos 2x}{2}\right) \\ &= \frac{5}{8}(\cos 2z - 1) + \frac{z\sin 2z}{4} + \frac{z^2}{4} + \int_0^z dx \frac{1-\cos 2x}{2x} \\ I_3 &= -\frac{1}{2}\int_0^z dx \left(\frac{1-\cos 2x}{2} - x\sin 2x + x^2\frac{1+\cos 2x}{2}\right)\frac{d}{dx}\frac{1}{x^2} \\ &= -\frac{1}{2z^2}\left(\frac{1-\cos 2z}{2} - z\sin 2z + z^2\frac{1+\cos 2z}{2}\right) + \int_0^z dx \frac{1}{2x}(1 - \cos 2x - x\sin 2x) \\ &= -\frac{1}{2z^2}\left(\frac{1-\cos 2z}{2} - z\sin 2z + z^2\right) + \int_0^z dx\frac{1-\cos 2x}{2x} \\ I_5 &= -\frac{1}{4}\int_0^z dx \left(\frac{1-\cos 2x}{2} - x\sin 2x + x^2\frac{1+\cos 2x}{2}\right)\frac{d}{dx}\frac{1}{x^4} \\ &= -\frac{1}{4z^4}\left(\frac{1-\cos 2z}{2} - z\sin 2z + z^2\frac{1+\cos 2z}{2}\right) + \int_0^z dx \frac{1}{4x^3}(1 - \cos 2x - x\sin 2x) \\ &= -\frac{1}{4z^4}\left(\frac{1-\cos 2z}{2} - z\sin 2z + z^2\right) + \frac{1}{8z}\sin 2z + \int_0^z dx\frac{1}{8x^2}(\sin 2x - 2x\cos x) \\ &= -\frac{1}{4z^4}\left(\frac{1-\cos 2z}{2} - z\sin 2z + z^2\right) + \frac{1}{4}\end{aligned}$$

Then

$$
\begin{aligned}
I(z) &= I_5 - \frac{2}{z^2} I_3 + \frac{2}{z^4} I_1 \\
&= \frac{1}{4} - \frac{1}{4z^4}\left(\frac{1-\cos 2z}{2} - z\sin 2z + z^2\right) + \frac{1}{z^4}\left(\frac{1-\cos 2z}{2} - z\sin 2z + z^2\right) \\
&\quad + \frac{2}{z^4}\left(\frac{5}{8}(\cos 2z - 1) + \frac{z\sin 2z}{4} + \frac{z^2}{4}\right) - \left(\frac{1}{z^2} - \frac{1}{z^4}\right)\int_0^z dx \frac{1-\cos 2x}{x} \\
&= \frac{1}{4} + \frac{5}{4z^2} - \frac{7}{8z^4}(1-\cos 2z) - \frac{1}{4z^3}\sin 2z - \left(\frac{1}{z^2} - \frac{1}{z^4}\right)\int_0^z dx \frac{1-\cos 2x}{x} \\
&= \frac{1}{4}\left[1 + \frac{5}{z^2} - \frac{7}{2z^4}(1-\cos 2z) - \frac{1}{z^3}\sin 2z - 4\left(\frac{1}{z^2} - \frac{1}{z^4}\right)\int_0^z dx \frac{1-\cos 2x}{x}\right]
\end{aligned}
$$

This exact evaluation of the integral goes to $1/4$ as $z \to \infty$ confirming the result quoted in part a).